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Mikhail V. Bondarko

To my wife Olga

ABSTRACT

The goal of this paper is to prove that if certain ‘standard’ conjectures on motives over algebraically closed fields hold, then over any ‘reasonable’ scheme S there exists a *motivic t -structure* for the category $\mathrm{DM}_c(S)$ of relative Voevodsky’s motives (to be more precise, for the Beilinson motives described by Cisinski and Deglise). If S is of finite type over a field, then the heart of this t -structure (the category of *mixed motivic sheaves* over S) is endowed with a *weight filtration* with semisimple factors. We also prove a certain ‘motivic decomposition theorem’ (assuming the conjectures mentioned) and characterize semisimple motivic sheaves over S in terms of those over its residue fields. Our main tool is the theory of *weight structures*. We actually prove somewhat more than the existence of a weight filtration for mixed motivic sheaves: we prove that the motivic t -structure is *transversal* to the Chow weight structure for $\mathrm{DM}_c(S)$ (that was introduced previously by Hébert and the author). We also deduce several properties of mixed motivic sheaves from this fact. Our reasoning relies on the degeneration of *Chow weight spectral sequences* for ‘perverse étale homology’ (which we prove unconditionally); this statement also yields the existence of the *Chow weight filtration* for such (co)homology that is strictly restricted by (‘motivic’) morphisms.

Introduction

The famous conjectures of Beilinson (see [Bei87, §5.10A]) predict the existence of an abelian category $\mathrm{MM}(S)$ of *mixed motivic sheaves* over a (more or less, arbitrary) scheme S . This category should be endowed with a so-called *weight filtration* whose factors are semisimple; it should possess an exact realization whose target is the category of perverse $(\mathbb{Q}_l\text{-})$ étale sheaves. The goal of this paper is to deduce these conjectures from certain ‘standard’ conjectures on motives over algebraically closed fields.

Let us explain this in more detail. It is widely believed that $\mathrm{MM}(S)$ should be the heart of a certain (motivic) t -structure for some triangulated category of (Voevodsky’s) motives over S . In this paper we treat this question for the category $\mathrm{DM}_c(S)$ of constructible Beilinson motives (as described in [CD09]) over a (*very reasonable*) base scheme S , and prove that a (‘nice’) motivic

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t -structure exists for it if it exists for Voevodsky's motives over algebraically closed fields. Recall that the latter assumption requires certain very hard 'standard' conjectures (especially for positive characteristic fields; see § 4.1 below and [Han99, § 2] for a discussion of those), yet it is nice to know that passing to relative motives in this matter conceals no additional difficulties. Note also that the paper [CH00] relies on the same conjectures that we need for our main results, whereas in [CH00] only the properties of (certain) 'pure' relative motivic sheaves (and only for S being a variety over a characteristic 0 field) are established. In particular, we prove a certain motivic version of the decomposition theorem for perverse sheaves (see § 4.2) that is much stronger than the corresponding result of [CH00]. We also characterize simple mixed motivic sheaves (those are certainly 'pure') in terms of those over the residue fields of S . Certainly, the results of [CD09] are crucial for our success here.

We now describe our central results in more detail, and also mention the main prerequisites for their proofs.

Our first principal result is as follows. Suppose that for some fixed prime l and for any universal domain K (of characteristic distinct from l ; a universal domain is an algebraically closed field of infinite transcendence degree over its prime subfield) there exists a t -structure t_{MM} for the category $\text{DM}_{\text{gm}}(K)$ of Voevodsky's motives over K that is strictly compatible with the (canonical) t -structure for \mathbb{Q}_l -adic étale sheaves (via étale homology; we call the heart of t_{MM} the category of mixed motives over K). Then t_{MM} also exists for motives over any 'reasonable' (see below) $\text{Spec } \mathbb{Z}[1/l]$ -scheme S . So, one may say that a certain $\text{MM}(S)$ exists in this case. The proof is quite easy (given the properties of Beilinson motives established in [CD09]); we just apply a simple gluing argument. Actually, it is not necessary to fix l here: if for each K of characteristic p there exists a (motivic) t -structure for $\text{DM}_{\text{gm}}(K)$ that is strictly compatible (as above) with $\mathbb{Q}_{l'}$ -adic étale (co)homology for any $l' \neq p$, then the motivic t -structure exists over any (reasonable) S (and it does not depend on l).

The second central result seems to be more interesting; its proof is more complicated. We verify that certain 'weights' exist for mixed motives over any *very reasonable* (see Definition 2.1.1(4) below) scheme S (in particular, if S is of finite type over a field); if $\text{char } S = 0$ or if 'the weights are nice' over a universal K such that $\text{char } K = \text{char } S$, then these weights are 'nice' over S also. This sentence requires a considerable amount of explanation, and we give it here.

The 'classical' approach to constructing weights for motives (originating from Beilinson) was to define a filtration for motives that would split Chow motives into their components corresponding to single (co)homology groups (i.e., would yield the so-called Chow–Kunneth decompositions). Since the existence of Chow–Kunneth decompositions is very much conjectural, it is no wonder that this approach has not yielded any significant (general) results up to this moment (to the knowledge of the author).

An alternative method for defining (certain) weights for motives was proposed and successfully implemented in [Bon10a]. To this end *weight structures* for triangulated categories were defined. This notion is a natural important counterpart of t -structures; somewhat similarly to t -structures, weight structures for a triangulated $\underline{\mathcal{C}}$ are defined in terms of $\underline{\mathcal{C}}_{w \leq 0}, \underline{\mathcal{C}}_{w \geq 0} \subset \text{Obj } \underline{\mathcal{C}}$. For the Chow weight structure w_{Chow} for $\text{DM}_c(S)$ its *heart* $\text{DM}_c(S)_{w_{\text{Chow}} \leq 0} \cap \text{DM}_c(S)_{w_{\text{Chow}} \geq 0}$ consists of Chow motives (over S ; these are 'ordinary' Chow motives if S is the spectrum of a perfect field); we avoid Chow–Kunneth decompositions this way. The weight structure w_{Chow} allows us to define certain spectral sequences for any (co)homology of motives; for singular and étale cohomology those are isomorphic to the 'classical' ones. The Chow weight structure w_{Chow} for $\text{DM}_c(S)$ was introduced in [Heb11] and [Bon14]; it is closely related to the weights for mixed complexes of sheaves

(as introduced in [BBD82, § 5.1.8]; see [Bon14, §§ 3.4–3.6]) and for mixed Hodge complexes and modules (see [Bon12, § 2.3]). All of these results are unconditional.

In [Bon12] and (especially) in the current paper we demonstrate that the Chow weight structure is also useful for the study of motivic conjectures. In particular (using the results of [Bon12, § 3]), we prove the following statement: if t_{MM} exists over a scheme S that is of finite type over a field, then Chow weight spectral sequences yield a weight filtration for S -motivic sheaves; this filtration is strictly respected by morphisms of motives. Our argument relies on the degeneration at E_2 of Chow weight spectral sequences for the ‘perverse étale homology’. We prove the latter result unconditionally. It also yields the existence of the Chow weight filtration for perverse étale (co)homology of motives that is strictly restricted by ‘motivic’ morphisms; so it could be useful in itself. The proof relies on certain new ‘continuity’ properties of the Chow weight structure.

Moreover, in [Bon12, § 1] also the conjectural relation of w_{Chow} to the motivic t -structure was axiomatized. The corresponding notion of *transversal* weight and t -structures was introduced, and several equivalent definitions of transversality were given. So we actually prove the following: if over a universal domain K of characteristic p (which could be 0) t_{MM} exists and is transversal to w_{Chow} , then the same is true for $\text{DM}_c(S)$ for any very reasonable S of characteristic p .

This ‘triangulated’ approach to weights (for mixed motives) has serious advantages over the (usual) ‘abelian’ version. First, it allows the conjectural properties of mixed motives to be combined with unconditional results on the Chow weight structure (and on Chow weight spectral sequences). We obtain some ‘new’ properties of mixed motivic sheaves this way; note that (by virtue of our results) all of them follow from ‘standard’ motivic conjectures (cf. the discussion in § 4.1 below). Besides, we obtain a description of weights for mixed motives whose only conjectural ingredient is the existence of t_{MM} . Finally, note that the ‘triangulated’ approach allows us to apply a certain gluing argument (heavily reliant on [BBD82, § 1.4]) that does not seem to work in the context of filtered abelian categories.

Summarizing: we prove that if a certain list of standard (motivic) conjectures over algebraically closed fields hold, then the category of mixed motivic sheaves exists over any reasonable scheme S ; for very reasonable S we obtain ‘nice weights’ for S -motivic sheaves. We also deduce (most of) the properties of this category that were conjectured by Beilinson and others, and prove some of their ‘triangulated extensions’. Besides, we prove a certain ‘motivic decomposition theorem’, and calculate the Grothendieck group of mixed motivic sheaves.

Finally, we note that the results of the current paper (as well as the results of [Bon14]) only rely upon a certain ‘axiomatization’ of Beilinson motives (i.e., on a certain list of their properties; cf. Remark 3.2.2(1) below). It follows that our arguments could be applied to the study of other categories satisfying similar properties. A natural candidate here would be Saito’s Hodge modules. It seems that this setting has already been thoroughly studied by Saito himself (cf. [Bon12, Proposition 2.3.1(I)]); on the other hand, our methods may possibly yield certain simplifications of his arguments.

We now list the contents of the paper. More details can be found at the beginning of each section.

Section 1 is dedicated to recalling some results from homological algebra. We recall some basics of t -structures. We also remind the reader of basic definitions and results on weight structures, weight filtrations and spectral sequences, as well as the notion of transversality of weight and t -structures (following [Bon10a] and [Bon12]). We also recall (mostly from [BBD82, § 1.4]) several basic results on gluing of t -structures and weight structures.

In § 2 we recall the basic properties of S -motives (as defined and studied in [CD09]) and the Chow weight structure for them (as introduced in [Heb11] and [Bon14]; we prove some new ‘continuity’ properties of the Chow weight structure). We also recall some properties of the perverse t -structure for \mathbb{Q}_l -étale sheaves, and study weight spectral sequences for the ‘perverse étale homology’. Those degenerate at E_2 if S is a very reasonable scheme; we conjecture that they degenerate for a general (reasonable) S also.

In § 3 we define the motivic t -structure (when it exists) as the one that is (strictly) compatible with the perverse t -structure for complexes of \mathbb{Q}_l -adic sheaves. We prove that the motivic t -structure exists over S if it exists over (all) universal domains. We also deduce some simple consequences from the ‘niceness’ of the motivic t -structure (i.e., from its transversality with w_{Chow}). These enable us to prove that over a very reasonable scheme there exists a nice t_l if the same is true over some universal domain of the same characteristic.

In § 4 we verify that the existence of a (nice) motivic t -structure and its independence from l follow from a certain list of (more or less) ‘standard’ motivic conjectures (over algebraically closed base fields). We also prove a certain ‘motivic decomposition theorem’ (modulo the conjectures mentioned). In particular, we characterize semisimple (pure) motives over S in terms of those over its residue fields. This enables us to calculate $K_0(\text{DM}_c(S))$.

Notation. \underline{C} below will always denote some triangulated category. t will always denote a bounded t -structure, and w will be a bounded weight structure (the theory of weight structures was thoroughly studied in [Bon10a]; see also § 1.1 below).

$D \subset \text{Obj } \underline{C}$ will be called extension-stable if for any distinguished triangle $A \rightarrow B \rightarrow C$ in \underline{C} we have $A, C \in D \implies B \in D$. Note that $\underline{C}^{t \leq i}$, $\underline{C}^{t \geq i}$, $\underline{C}^{t=0}$ (see § 1.2), $\underline{C}_{w \geq i}$, and $\underline{C}_{w \leq i}$ (see § 1.1) are extension-stable for any t, w and any $i \in \mathbb{Z}$.

For $D, E \subset \text{Obj } \underline{C}$ we will write $D \perp E$ if $\underline{C}(X, Y) = \{0\}$ for all $X \in D, Y \in E$. For $D \subset \underline{C}$ we will denote by D^\perp the class

$$\{Y \in \text{Obj } \underline{C} : X \perp Y \ \forall X \in D\}.$$

Dually, ${}^\perp D$ is the class $\{Y \in \text{Obj } \underline{C} : Y \perp X \ \forall X \in D\}$.

A full subcategory $B \subset \underline{C}$ is called *Karoubi-closed* in \underline{C} if B contains all \underline{C} -retracts of its objects. For $B \subset \underline{C}$ we will call the subcategory of \underline{C} whose objects are all retracts of objects of B (in \underline{C}) the *Karoubi-closure* of B in \underline{C} .

For a class of objects $C_i \in \text{Obj } \underline{C}$, $i \in I$, we will denote by $\langle C_i \rangle$ the smallest strictly full triangulated subcategory containing all C_i . We will call the Karoubi-closure of $\langle C_i \rangle$ in \underline{C} the *triangulated category generated by C_i* .

\underline{A} will always be an abelian category. We will call a covariant (respectively contravariant) additive functor $H : \underline{C} \rightarrow \underline{A}$ *homological* (respectively *cohomological*) if it converts distinguished triangles into long exact sequences.

1. Preliminaries on triangulated categories, weight structures and t -structures

In § 1.1 we recall some basics on weight structures (as developed in [Bon10a]).

In § 1.2 we recall the definition of a t -structure and introduce some notation.

In § 1.3 we study weight spectral sequences (following [Bon10a, § 2] and [Bon12, § 3]), their degeneration, and weight filtrations for $\underline{H}t$ coming from w .

In § 1.4 we recall the notion of transversal weight structures and t -structures (as introduced in [Bon12]).

In § 1.5 we prove (relying heavily upon [BBD82, § 1.4]) several auxiliary statements on t -structures and weights in the ‘gluing setting’.

1.1 Weight structures: short reminder

DEFINITION 1.1.1. (I) A pair of subclasses $\underline{C}_{w \leq 0}, \underline{C}_{w \geq 0} \subset \text{Obj } \underline{C}$ will be said to define a weight structure w for \underline{C} if they satisfy the following conditions:

- (i) $\underline{C}_{w \geq 0}, \underline{C}_{w \leq 0}$ are additive and Karoubi-closed in \underline{C} (i.e., contain all \underline{C} -retracts of their objects);
- (ii) *semi-invariance with respect to translations*, $\underline{C}_{w \leq 0} \subset \underline{C}_{w \leq 0}[1]$, $\underline{C}_{w \geq 0}[1] \subset \underline{C}_{w \geq 0}$;
- (iii) *orthogonality*, $\underline{C}_{w \leq 0} \perp \underline{C}_{w \geq 0}[1]$;
- (iv) *weight decompositions*.

For any $M \in \text{Obj } \underline{C}$ there exists a distinguished triangle

$$B \rightarrow M \rightarrow A \xrightarrow{f} B[1] \quad (1)$$

such that $A \in \underline{C}_{w \geq 0}[1]$, $B \in \underline{C}_{w \leq 0}$.

(II) The category $\underline{Hw} \subset \underline{C}$ whose objects are $\underline{C}_{w=0} = \underline{C}_{w \geq 0} \cap \underline{C}_{w \leq 0}$, $\underline{Hw}(Z, T) = \underline{C}(Z, T)$, for $Z, T \in \underline{C}_{w=0}$, will be called the *heart* of w .

(III) $\underline{C}_{w \geq i}$ (respectively $\underline{C}_{w \leq i}$, respectively $\underline{C}_{w=i}$) will denote $\underline{C}_{w \geq 0}[i]$ (respectively $\underline{C}_{w \leq 0}[i]$, respectively $\underline{C}_{w=0}[i]$).

(IV) We will say that (\underline{C}, w) is *bounded* if $\bigcup_{i \in \mathbb{Z}} \underline{C}_{w \leq i} = \text{Obj } \underline{C} = \bigcup_{i \in \mathbb{Z}} \underline{C}_{w \geq i}$.

(V) Let \underline{C} and \underline{C}' be triangulated categories endowed with weight structures w and w' , respectively; let $F : \underline{C} \rightarrow \underline{C}'$ be an exact functor. F will be called *left weight-exact* (with respect to w, w') if it maps $\underline{C}_{w \leq 0}$ to $\underline{C}'_{w' \leq 0}$; it will be called *right weight-exact* if it maps $\underline{C}_{w \geq 0}$ to $\underline{C}'_{w' \geq 0}$. F is called *weight-exact* if it is both left and right weight-exact.

Remark 1.1.2. (1) A simple (and yet useful) example of a weight structure comes from the stupid filtration on the homotopy categories $K(B) \supset K^b(B)$ of cohomological complexes for an arbitrary additive category B . In this case $K(B)_{w \leq 0}$ (respectively $K(B)_{w \geq 0}$) is the class of complexes that are homotopy equivalent to complexes concentrated in degrees greater than or equal to 0 (respectively less than or equal to 0). The heart of this weight structure (either for $K(B)$ or for $K^b(B)$) is the Karoubi-closure of B in the corresponding category.

(2) A weight decomposition (of any $M \in \text{Obj } \underline{C}$) is (almost) never canonical. Yet for an $m \in \mathbb{Z}$ we will often need an (arbitrary) choice of a weight decomposition of $M[-m]$ shifted by $[m]$. This way we obtain a distinguished triangle

$$w_{\leq m} M \rightarrow M \rightarrow w_{\geq m+1} M \quad (2)$$

with some $w_{\geq m+1} M \in \underline{C}_{w \geq m+1}$, $w_{\leq m} M \in \underline{C}_{w \leq m}$ (see [Bon10a, Remark 1.2.2]); we will use this notation below (though $w_{\geq m+1} M$ and $w_{\leq m} M$ are not canonically determined by M unless we impose some additional restrictions on these objects).

(3) *Caution on signs of weights.* When the author defined weight structures (in [Bon10a]), he considered $(\underline{C}^{w \leq 0}, \underline{C}^{w \geq 0})$ such that $\underline{C}^{w \leq 0}$ is stable with respect to $[1]$ (similarly to the usual convention for t -structures); in particular, this meant that for $\underline{C} = K(B)$ and for the ‘stupid’ weight structure for it mentioned above a complex C whose only non-zero term is the fifth one (i.e., $C^5 \neq 0$) was ‘of weight 5’. Whereas this (*cohomological*) convention seems to be quite natural, for weights of mixed Hodge complexes, mixed Hodge modules (see [Bon12, Proposition 2.6]),

and mixed complexes of sheaves (see [Bon14, Proposition 3.6.1] and Proposition 2.5.1(I) below) ‘classically’ exactly the opposite convention was used (so, in this convention our \underline{C} is of weight -5). For this reason, in the current paper we use the ‘reverse’ (*homological*) convention for the signs of weights, that is compatible with the ‘classical weights’ (this convention for the Chow weight structure for motives was used in [Heb11, Bon12, Bon14]; see also [Bon11]); so the signs of weights used below will be opposite to those in [Bon10a, Bon10b].

We now recall those properties of weight structures that will be needed below.

PROPOSITION 1.1.3. *Let \underline{C} be a triangulated category. w will be a weight structure for \underline{C} everywhere except in assertion (i).*

(i) (C_1, C_2) ($C_1, C_2 \subset \text{Obj } \underline{C}$) define a weight structure for \underline{C} if and only if (C_2^{op}, C_1^{op}) define a weight structure for \underline{C}^{op} .

(ii) $\underline{C}_{w \leq 0}$, $\underline{C}_{w \geq 0}$, and $\underline{C}_{w=0}$ are extension-stable.

(iii) Let w be a weight structure for \underline{C} . Then $\underline{C}_{w \geq 0} = (\underline{C}_{w \leq -1})^\perp$ and $\underline{C}_{w \leq 0} = {}^\perp \underline{C}_{w \geq 1}$ (see Notation).

(iv) Let \underline{C} and \underline{D} be triangulated categories endowed with weight structures w and v , respectively; let w be bounded. Then an exact functor $F : \underline{C} \rightarrow \underline{D}$ is left (respectively right) weight-exact if and only if $F(\underline{C}_{w=0}) \subset \underline{D}_{v \leq 0}$ (respectively $F(\underline{C}_{w=0}) \subset \underline{D}_{v \geq 0}$).

(v) Suppose that v is another weight structure for \underline{C} ; let $\underline{C}_{v \leq 0} \subset \underline{C}_{w \leq 0}$ and $\underline{C}_{v \geq 0} \subset \underline{C}_{w \geq 0}$. Then $v = w$ (i.e., the inclusions are equalities).

(vi) If w is bounded, then $\underline{C}_{w \leq 0}$ is the smallest extension-stable subclass of $\text{Obj } \underline{C}$ containing $\bigcup_{i \leq 0} \underline{C}_{w=i}$; $\underline{C}_{w \geq 0}$ is the smallest extension-stable class of $\text{Obj } \underline{C}$ containing $\bigcup_{i \geq 0} \underline{C}_{w=i}$.

Proof. Most of the assertions were proved in [Bon10a] (pay attention to Remark 1.1.2(3)!); see [Bon14, Proposition 1.2.3] for more details. \square

1.2 t -structures: a very short reminder and notation

To fix the notation we recall the definition of a t -structure (following [BBD82]).

DEFINITION 1.2.1. A pair of subclasses $\underline{C}^{t \geq 0}, \underline{C}^{t \leq 0} \subset \text{Obj } \underline{C}$ will be said to define a t -structure t if they satisfy the following conditions:

- (i) $\underline{C}^{t \geq 0}, \underline{C}^{t \leq 0}$ are strict, i.e., contain all objects of \underline{C} isomorphic to their elements;
- (ii) $\underline{C}^{t \geq 0} \subset \underline{C}^{t \geq 0}[1]$, $\underline{C}^{t \leq 0}[1] \subset \underline{C}^{t \leq 0}$;
- (iii) *orthogonality*, $\underline{C}^{t \leq 0}[1] \perp \underline{C}^{t \geq 0}$;
- (iv) *t -decompositions*.

For any $M \in \text{Obj } \underline{C}$ there exists a distinguished triangle

$$A \rightarrow M \rightarrow B \rightarrow A[1] \quad (3)$$

such that $A \in \underline{C}^{t \leq 0}$, $B \in \underline{C}^{t \geq 0}[-1]$.

Bounded t -structures can be defined similarly to Definition 1.1.1(IV). We recall that all the weight structures and t -structures that we will consider below will be bounded.

We will need some more notation for t -structures.

DEFINITION 1.2.2. (1) The category \underline{Ht} whose objects are $\underline{C}^{t=0} = \underline{C}^{t \geq 0} \cap \underline{C}^{t \leq 0}$, $\underline{Ht}(M, Y) = \underline{C}(M, Y)$ for $M, Y \in \underline{C}^{t=0}$, will be called the *heart* of t . Recall that \underline{Ht} is always abelian; short exact sequences in \underline{Ht} come from distinguished triangles in \underline{C} .

(2) $\underline{C}^{t \geq l}$ (respectively $\underline{C}^{t \leq l}$) will denote $\underline{C}^{t \geq 0}[-l]$ (respectively $\underline{C}^{t \leq 0}[-l]$).

Remark 1.2.3. (1) Recall that (3) defines additive functors $\underline{C} \rightarrow \underline{C}^{t \leq 0} : M \mapsto A$ and $C \rightarrow \underline{C}^{t \geq 1} : M \mapsto B$. We will denote A, B by $M^{t \leq 0}$ and $M^{t \geq 1}[-1]$, respectively. (3) will be called the *t-decomposition* of M .

More generally, the t -components of $M[i]$ (for any $i \in \mathbb{Z}$) will be denoted by $M^{t \leq i} \in \underline{C}^{t \leq 0}$ and $M^{t \geq i+1}[-1] \in \underline{C}^{t \geq 1}$, respectively.

$t_{\leq i}M$ will denote $M^{t \leq i}[-i]$; $t_{\geq i}M$ will denote $M^{t \geq i}[-i]$.

(2) The functor $M \mapsto t_{\geq 0}M$ is left adjoint to the inclusion $\underline{C}^{t \geq 0} \rightarrow \underline{C}$.

(3) We will also need the following easy (and well-known) properties of t -structures.

Firstly [BBD82, Proposition 1.3.17(iii)], if a functor F is left adjoint to G , and their targets are endowed with t -structures, then F is right t -exact if and only if G is left t -exact. The latter assertions mean that F respects ‘ t -negative’ objects, whereas G respects t -positive ones.

Secondly, if for two t -structures t and t' on a triangulated category \underline{C} the identity functor is t -exact (for the pairs (\underline{C}, t) and (\underline{C}, t') , i.e., $\underline{C}^{t \leq 0} \subset \underline{C}^{t' \leq 0}$ and $\underline{C}^{t \geq 0} \subset \underline{C}^{t' \geq 0}$), then $t = t'$. Indeed, the previous statement yields that the identity is also t -exact as a functor from (\underline{C}, t') to (\underline{C}, t) .

(4) We denote by H_0^t the zeroth homology functor corresponding to t . Shifting the t -decomposition of $M^{t \leq 0}[-1]$ by $[1]$ we obtain a canonical and functorial (with respect to M) distinguished triangle $t_{\leq -1}M \rightarrow t_{\leq 0}M \rightarrow H_0^t(M)$; we denote $H_0^t(M[i])$ by $H_i^t(M)$.

If a t -structure is non-degenerate (i.e., if $\bigcap_{i \in \mathbb{Z}} \underline{C}^{t \leq i} = \bigcap_{i \in \mathbb{Z}} \underline{C}^{t \geq i} = \{0\}$; note that this is certainly the case for bounded t -structures) then the collection of H_i^t is conservative. Moreover, in this case for $C \in \text{Obj } \underline{C}$ we have $C \in \underline{C}^{t \leq 0}$ (respectively $C \in \underline{C}^{t \geq 0}$) whenever $H_i^t(C) = 0$ for all $i > 0$ (respectively for all $i < 0$).

1.3 On weight filtrations and (degenerating) weight spectral sequences

We recall certain properties of weight filtrations and weight spectral sequences. Most of them were established in [Bon10a, § 2], whereas the degeneration of weight spectral sequences was studied in [Bon12, § 3].

Let \underline{A} be an abelian category. In [Bon10a, § 2] for $H : \underline{C} \rightarrow \underline{A}$ that is either cohomological or homological (i.e., it is either covariant or contravariant, and converts distinguished triangles into long exact sequences) certain *weight filtrations* and *weight spectral sequences* (corresponding to w) were introduced. Below we will be more interested in the homological functor case; certainly, one can pass to cohomology by a simple reversion of arrows (cf. [Bon10a, § 2.4]).

DEFINITION 1.3.1. Let $H : \underline{C} \rightarrow \underline{A}$ be a covariant functor, $i \in \mathbb{Z}$.

(1) We denote $H \circ [i] : \underline{C} \rightarrow \underline{A}$ by H_i .

(2) We choose some $w_{\leq i}M$ and define the *weight filtration* for H by $W_i H : M \mapsto \text{Im}(H(w_{\leq i}M) \rightarrow H(M))$.

Recall that $W_i H$ is functorial in M (in particular, it does not depend on the choice of $w_{\leq i}M$); see [Bon10a, Proposition 2.1.2(1)].

We now recall some of the properties of weight spectral sequences; we are especially interested in the case where they degenerate.

PROPOSITION 1.3.2. (I) For a homological functor $H : \underline{C} \rightarrow \underline{A}$ and any $M \in \text{Obj } \underline{C}$ there exists a spectral sequence $T = T_w(H, M)$ with $E_1^{pq}(T) = H_q(M^p)$ for certain $M^m \in \underline{C}_{w=0}$ (coming from certain weight decompositions as in (2)) that converges to $E_\infty^{p+q} = H_{p+q}(M)$. T is \underline{C} -functorial in M starting from E_2 . Also, if $F : \underline{A} \rightarrow \underline{A}'$ is an exact functor of abelian categories then applying F to (a choice of) $T_w(H, M)$, one obtains a choice of $T_w(F \circ H, M)$ coming from the same choices of weight decompositions; this gives a canonical isomorphism starting from E_2 . Besides, the filtration step given by $(E_\infty^{l, m-l} : l \geq k)$ on $H_m(M)$ (for some $k, m \in \mathbb{Z}$) equals $(W_{-k}H_m)(M)$.

Moreover, $T(H, M)$ comes from an exact couple with $D_1^{pq} = H_{p+q}(w_{\leq -p}M)$ (here one can fix any choice of $w_{\leq -p}M$).

We will say that T degenerates at E_2 (for a fixed H) if $T_w(H, M)$ does so for any $M \in \text{Obj } \underline{C}$.

(II) Suppose that T degenerates at E_2 (as above), $i \in \mathbb{Z}$. Then the following statements are fulfilled.

- (1) The functors $W_i H$ and $W'_i H : M \mapsto H(M)/W_{i-1}H(M)$ are homological.
- (2) For any $f \in \underline{C}(M, Y)$ the morphism $H(f)$ is strictly compatible with the filtration of H by W_i , i.e., $W_i H(M)$ surjects onto $W_i H(Y) \cap \text{Im } H(f)$.

Proof. This is immediate from [Bon12, Proposition 3.1.2] (recall that w is bounded by our convention). \square

Remark 1.3.3. The description of the exact couple for $T_w(H, M)$ (which can be found in [Bon12, Proposition 3.1.2]) also easily yields the following results.

(1) If $H' = H \circ [d]$ then $T_w(H', M)$ for any $M \in \text{Obj } \underline{C}$ can be obtained by a ‘shift’ from $T_w(H, M)$, i.e., one can take $E_r^{p,q} T_w(H', M) = E_r^{p,q+d} T_w(H, M)$ for all $r \geq 1, p, q \in \mathbb{Z}$, and the differentials behave in the similar way. Certainly, this yields a functorial description of $T_w(H', M)$ starting from E_2 .

(2) The functoriality of $T_w(H, M)$ in H described in assertion (I) of the proposition can be easily generalized as follows. Let

$$\begin{array}{ccc} \underline{C} & \xrightarrow{F} & \underline{C}' \\ \downarrow H & & \downarrow H' \\ \underline{A} & \xrightarrow{G} & \underline{A}' \end{array} \quad (4)$$

be a commutative square of functors, where \underline{C}' is a triangulated category endowed with a weight structure w' such that F is a weight-exact (and exact) functor, and G is an exact functor of abelian categories. Then for any $M \in \text{Obj } \underline{C}$ one can obtain $T_{w'}(H', F(M))$ by applying G (termwise) to $T_w(H, M)$. Again, this is a functorial isomorphism starting from E_2 .

(3) Hence in this setting for any $d \in \mathbb{Z}$ we obtain the following: if the spectral sequence $T_w(H, M)$ degenerates at E_2 , then $T_{w'}(H' \circ [d], F(M))$ also does. The converse statement is also true if G is conservative.

This is a generalization of Proposition 1.3.6(II) below.

We now introduce the notion of a weight filtration for an abelian category following [BK14, Definition D.7.2].

DEFINITION 1.3.4. For an abelian \underline{A} , we will say that an increasing family of full subcategories $\underline{A}_{\leq i} \subset \underline{A}$, $i \in \mathbb{Z}$, yields a *weight filtration* for \underline{A} if $\bigcap_{i \in \mathbb{Z}} \underline{A}_{\leq i} = \{0\}$, $\bigcup_{i \in \mathbb{Z}} \underline{A}_{\leq i} = \underline{A}$, and there exist exact right adjoints $W_{\leq i}$ to the embeddings $\underline{A}_{\leq i} \rightarrow \underline{A}$.

We will need the following statement.

LEMMA 1.3.5. Consider a family $\underline{A}_{\leq m}$, $m \in \mathbb{Z}$, yielding a weight filtration for \underline{A} . Then the following statements are valid.

- (1) $\underline{A}_{\leq m}$ are exact abelian subcategories of \underline{A} .
- (2) All $W_{\leq m}$ are idempotent endofunctors.
- (3) The adjunctions yield functorial embeddings of $W_{\leq m}M \rightarrow M$ such that $W_{\leq m-1}M \subset W_{\leq m}M$ for all $m \in \mathbb{Z}$, and the functors $W_{\geq m} : M \mapsto M/W_{\leq m-1}M$ are exact also.

(4) The categories \underline{A}_m being the ‘kernels’ of the restriction of $W_{\leq m-1}$ to $\underline{A}_{\leq m}$, are abelian, and $\underline{A}_m \perp \underline{A}_j$ for any $j \neq m$.

Proof. As was said in the proof of [Bon12, Lemma 2.1.2], the result easily follows from [BK14, Remark D.7.8, Lemma D.7.5, and Proposition D.7.4(4)]. \square

We now fix certain (bounded) w and t for \underline{C} , and study a condition ensuring that w induces a weight filtration for $\underline{H}t$.

PROPOSITION 1.3.6. *Let $H = H_0^t$.*

(I) *Suppose that the corresponding spectral sequence T degenerates. Then the functors $W_i H : \underline{C} \rightarrow \underline{H}t$ are homological. The restrictions $W_{\leq i}$ of $W_i H$ to $\underline{H}t$ define a weight filtration for this category. Besides, $W_i H = W_{\leq i} \circ H$.*

(II) *Let \underline{B} be an abelian category; let $F : \underline{H}t \rightarrow \underline{B}$ be an exact functor.*

(1) *Suppose that T degenerates. Then $T_w(F \circ H, -)$ also does.*

(2) *Conversely, suppose that F is conservative and that $T_w(F \circ H, -)$ degenerates. Then T degenerates.*

Moreover, for $M \in \underline{C}^{t=0}$ we have $W_{\leq i} M = M$ (respectively $W_{\leq i} M = 0$) if and only if $W_i(F \circ H)(M) = F(M)$ (respectively $W_i(F \circ H)(M) = 0$).

Proof. This is (part of) [Bon12, Proposition 3.2.1]. \square

1.4 On transversal weight and t -structures

Let t be a t -structure for \underline{C} , and w be a weight structure for it.

DEFINITION 1.4.1. (1) For some \underline{C}, t, w we will say that a distinguished triangle (2) (for some m, M) is *nice* if $w_{\leq m} M, M, w_{\geq m+1} M \in \underline{C}^{t=0}$. We will also say that this distinguished triangle is a *nice decomposition* of M (for the corresponding m).

(2) Suppose (as we always do) that t and w are bounded. We will say that t and w are transversal if a nice decomposition exists for any $m \in \mathbb{Z}$ and any $M \in \underline{C}^{t=0}$.

PROPOSITION 1.4.2. (I) We fix some \underline{C}, w, t, m . Suppose that for a certain $N \subset \underline{C}^{t=0}$ a nice decomposition exists for any $M \in N$. Consider $N' \subset \underline{C}^{t=0}$, being the smallest subclass containing N that satisfies the following condition: if $A, C \in N'$,

$$A \xrightarrow{f} B \xrightarrow{g} C \quad (5)$$

is a complex (i.e., $g \circ f = 0$), f is monomorphic, g is epimorphic, $\text{Ker } g / \text{Im } f \in N'$, then $B \in N'$. Then a nice choice of (2) exists for any $M \in N'$.

(II) If t is transversal to w , then the following statements are fulfilled for any $i \in \mathbb{Z}$, $M \in \underline{C}^{t=0}$, $Y \in \text{Obj } \underline{C}$.

(1) For any H that could be presented as $F \circ H_m^t$, where $F : \underline{H}t \rightarrow \underline{A}$ is an exact functor, the weight spectral sequence $T_w(H, -)$ degenerates at E_2 .

(2) Nice decompositions exist and are $\underline{H}t$ -functorial in M (for a fixed m). The corresponding functor $W_{\leq m} : M \mapsto w_{\leq m} M$ can be described as (the restriction to $\underline{H}t$ of) $W_m H_0^t$ (see Definition 1.3.1(2)); i.e., it coincides with the functor $W_{\leq m}$ given by Proposition 1.3.6(I).

(3) The category $\underline{A}_m = \underline{C}^{t=0} \cap \underline{C}_{w=m}$ is (abelian) semisimple; there is a splitting $\underline{C}_{w=0} = \bigoplus_{m \in \mathbb{Z}} \text{Obj } \underline{A}_m[-m]$ given by $Y \mapsto \bigoplus_{i \in \mathbb{Z}} H_i^t(Y)[-i]$.

- (4) $W_{\leq m}M$ yield an increasing filtration for M whose m th factor belongs to \underline{A}_m . Moreover, this filtration is uniquely and functorially determined by this condition.
- (5) $Y \in \underline{C}_{w \leq m}$ (respectively $Y \in \underline{C}_{w \geq m}$) if and only if for any $j \in \mathbb{Z}$ we have $W_{\leq m+j}(H_j^t(Y)) = H_j^t(Y)$ (respectively $W_{\leq m+j-1}(H_j^t(Y)) = 0$).

(III) For any $t, w, m \in \mathbb{Z}$, and any choice of $w_{\leq m}M$ (and a morphism $w_{\leq m}M \rightarrow M$ corresponding to a weight decomposition) consider the morphism $f_m(M) : (W_m H_0^t)(M) \rightarrow M$ (cf. Definition 1.3.1(2)). Then t is transversal to w if and only if this morphism extends to a nice decomposition (for any M, m).

(IV) For a family of semisimple (abelian) $\{\underline{A}_m \subset \underline{C}, m \in \mathbb{Z}\}$, suppose that $\langle \bigcup_{m \in \mathbb{Z}} \text{Obj } \underline{A}_m \rangle = \underline{C}$, and $\underline{A}_m \perp \underline{A}_j[s]$ for any $m, j, s \in \mathbb{Z}$ such that either $s < 0$, or $s > m - j$, or $s = 0$ and $m > j$. Then there exist transversal w and t such that $\underline{C}_{w=0} = \bigoplus_m \text{Obj } \underline{A}_m[-m]$, and $\underline{H}t$ is the smallest extension-stable subcategory of \underline{C} containing $\bigcup \underline{A}_m$.

(V) For any t, w , t is transversal to w if and only if there exists a family of semisimple abelian categories $\underline{A}_m \subset \underline{H}t$ ($m \in \mathbb{Z}$) such that: $\text{Obj } \underline{A}_m \cap \text{Obj } \underline{A}_j = \{0\}$ for all $j \neq m$, $j \in \mathbb{Z}$, and $\underline{C}_{w=0} = \bigoplus_m \text{Obj } \underline{A}_m[-m]$.

Proof. (I) This is [Bon12, Lemma 1.1.3].

(II) (1) This is immediate from [Bon12, Proposition 3.2.1(II, III.1)].

(2) The functoriality of nice decompositions is the condition (iv') of [Bon12, Theorem 1.2.1] (which is equivalent to condition (iv) of [Bon12, Theorem 1.2.1] that we took above for the definition of transversality). The equality of two distinct descriptions of $W_{\leq m}$ is given by [Bon12, Proposition 3.2.1(II)].

(3) See [Bon12, Remark 1.2.3(2)].

(4) This is immediate from condition (iii') of [Bon12, Theorem 1.2.1] (cf. the proof of assertion II.2).

(5) This is [Bon12, Proposition 1.2.4(I.2)].

(III) See [Bon12, Remark 1.2.3(4)].

(IV) [Bon12, Theorem 1.2.1] implies that such a family of \underline{A}_m does yield some transversal structures t, w . Besides, [Bon12, Theorem 1.2.1] also allows us to calculate $\underline{H}t$, whereas $\underline{C}_{w=0} = \bigoplus_m \text{Obj } \underline{A}_m[-m]$ by [Bon12, Remark 1.8(2)].

(V) Since $\langle \underline{H}w \rangle = \underline{C}$, we obtain $\langle \bigcup_{m \in \mathbb{Z}} \text{Obj } \underline{A}_m \rangle = \underline{C}$. Since \underline{A}_m are semisimple, we obtain that $\underline{A}_m \perp \underline{A}_j$ for any $m \neq j$. The orthogonality axioms of weight structures and t -structures also yield the remaining orthogonality conditions that are needed in order to apply the previous assertion. Hence, certain transversal t' and w' exist; besides, $\underline{H}w' = \underline{H}w$ and $\underline{H}t' \subset \underline{H}t$. Since w' is bounded, this implies $w = w'$ (see Proposition 1.1.3(iv), (v)). Since t' is bounded also, we easily deduce that $t' = t$. \square

Remark 1.4.3. In the case of motives (of smooth projective varieties over a field) the splittings mentioned in assertions (II)(3) and (IV) correspond to the so-called *Chow–Kunneth* decompositions ('of the diagonal').

1.5 Some auxiliary 'gluing statements'

Below we will apply several gluing arguments. We chose to gather the definitions and auxiliary statements related to this matter here.

DEFINITION 1.5.1. (1) The 9-tuple $(\underline{C}, \underline{D}, \underline{E}, i_*, j^*, i^*, i^!, j_!, j_*)$ is called a *gluing datum* if it satisfies the following conditions:

- (i) $\underline{C}, \underline{D}, \underline{E}$ are triangulated categories; $i_* : \underline{D} \rightarrow \underline{C}$, $j^* : \underline{C} \rightarrow \underline{E}$, $i^* : \underline{C} \rightarrow \underline{D}$, $i^! : \underline{C} \rightarrow \underline{D}$, $j_* : \underline{E} \rightarrow \underline{C}$, $j_! : \underline{E} \rightarrow \underline{C}$ are exact functors;
- (ii) i^* (respectively $i^!$) is left (respectively right) adjoint to i_* ; $j_!$ (respectively j_*) is left (respectively right) adjoint to j^* ;
- (iii) i_* is a full embedding; j^* is isomorphic to the localization (functor) of \underline{C} by $i_*(\underline{D})$;
- (iv) for any $M \in \text{Obj } \underline{C}$ the pairs of morphisms $j_! j^* M \rightarrow M \rightarrow i_* i^* M$ and $i_* i^! M \rightarrow M \rightarrow j_* j^* M$ can be completed to distinguished triangles (here the morphisms come from the adjunctions of (ii));
- (v) $i^* j_! = 0$; $i^! j_* = 0$;
- (vi) all of the adjunction transformations $i^* i_* \rightarrow \text{id}_{\underline{D}} \rightarrow i^! i_*$ and $j^* j_* \rightarrow \text{id}_{\underline{E}} \rightarrow j^* j_!$ are isomorphisms of functors.

(2) In the setting of part (1) of this definition, we will say that $M \in \text{Obj } \underline{C}$ is a *lift* of $Y \in \text{Obj } \underline{E}$ if $j^* M \cong Y$. Similarly, a lift of a distinguished triangle C in \underline{E} is a distinguished triangle C' in \underline{C} such that $j^* C' \cong C$.

(3) In the setting of part (1), suppose that \underline{C} is endowed with a t -structure $t = t_{\underline{C}}$. We define the *intermediate image* functor $j_{!*} : \underline{E} \rightarrow \underline{C}$ as $M \mapsto \text{Im}(H_0^{t_{\underline{C}}} j_! M \rightarrow H_0^{t_{\underline{C}}} j_* M)$; here the morphism $j_! \rightarrow j_*$ comes from adjunction (and we use the fact that $j^* j_! \cong j^* j_* \cong 1_{\underline{E}}$; cf. [BBD82, Equation (1.4.6.2) and Definition 1.4.22]).

(4) In the setting of part (1), suppose also that \underline{D} and \underline{E} are endowed with certain t -structures $t_{\underline{D}}$ and $t_{\underline{E}}$, respectively. Then we will say that a t -structure $t = t_{\underline{C}}$ for \underline{C} is *glued from* $t_{\underline{D}}$ and $t_{\underline{E}}$ if we have $\underline{C}^{t_{\underline{C}} \leq 0} = \{M \in \text{Obj } \underline{C} : j^* M \in \underline{E}^{t_{\underline{E}} \leq 0} \text{ and } i^* M \in \underline{D}^{t_{\underline{D}} \leq 0}\}$ and $\underline{C}^{t_{\underline{C}} \geq 0} = \{M \in \text{Obj } \underline{C} : j^* M \in \underline{E}^{t_{\underline{E}} \geq 0} \text{ and } i^! M \in \underline{D}^{t_{\underline{D}} \geq 0}\}$. In this case we will also say that \underline{C} , \underline{D} , and \underline{E} are *endowed with compatible t -structures*.

Remark 1.5.2. Our definition of a gluing datum is far from being the ‘minimal’ one. Actually, it is well known (see [Nee01, ch. 9]) that a gluing datum can be uniquely recovered from an inclusion $\underline{D} \rightarrow \underline{C}$ of triangulated categories that admits both a left and a right adjoint functor.

Our notation for a gluing datum is (certainly) coherent with Proposition 2.2.1(viii) below.

PROPOSITION 1.5.3. (I) In the setting of Definition 1.5.1(1) assume that \underline{D} is endowed with a t -structure $t_{\underline{D}}$. Then for any $M \in \text{Obj } \underline{C}$ any distinguished triangle $A' \rightarrow M' (= j^* M) \rightarrow B'$ (in \underline{E}) possesses a lift $A \rightarrow M \rightarrow B$ (see Definition 1.5.1(2)) such that $i^* A \in \underline{D}^{t_{\underline{D}} \leq 0}$ and $i^! B \in \underline{D}^{t_{\underline{D}} \geq 1}$.

(II) In the setting of Definition 1.5.1(4) the following statements are fulfilled:

- (1) there exists a t -structure $t_{\underline{C}}$ for \underline{C} glued from $t_{\underline{D}}$ and $t_{\underline{E}}$;
- (2) $t_{\underline{C}}$ is characterized by the following property: i_* and j^* are t -exact.

Moreover, $j_!$ and i^* are right t -exact (see Remark 1.2.3(3)), whereas j_* and $i^!$ are left t -exact (with respect to $t_{\underline{D}}$, $t_{\underline{C}}$, and $t_{\underline{E}}$, respectively).

(III) In the setting of Definition 1.5.1(1) assume that $\underline{C}, \underline{D}, \underline{E}$ are endowed with weight structures $w_{\underline{C}}, w_{\underline{D}}$, and $w_{\underline{E}}$, respectively, and that i_* and j^* are weight-exact. Then we will say that $w_{\underline{C}}, w_{\underline{D}}$, and $w_{\underline{E}}$ are *compatible*.

In this situation $j_!$ and i^* are left weight-exact, whereas j_* and $i^!$ are right weight-exact. Besides, we have $\underline{C}_{w \geq 0} = \{M \in \text{Obj } \underline{C} : i^! M \in \underline{D}_{w_{\underline{D}} \geq 0}, j^* M \in \underline{E}_{w_{\underline{E}} \geq 0}\}$ and $\underline{C}_{w \leq 0} = \{M \in \text{Obj } \underline{C} : i^* M \in \underline{D}_{w_{\underline{D}} \leq 0}, j^* M \in \underline{E}_{w_{\underline{E}} \leq 0}\}$.

(IV) Assume that \underline{C} , \underline{D} , and \underline{E} are endowed with compatible t -structures (see Definition 1.5.1(4)). Then for any $M, Y \in \underline{C}^{t=0}$, $M', Y' \in \underline{E}^{t=0}$, and $i \in \mathbb{Z}$, the following statements are valid:

- (i) $j^* j_! M' \cong j^* j_{!*} M' \cong j^* j_* M' \cong M'$;
- (ii) $i^* j_{!*} M' \cong t_{\leq -1} i^* j_* M'$ and $i^! j_{!*} M' \cong t_{\geq 1} i^! j_! M'$;
- (iii) if a complex $A \rightarrow B \rightarrow C$ (in $\underline{Ht}_{\underline{D}}$) is exact in the term B , then the middle-term homology object of the complex $j_{!*} A \rightarrow j_{!*} B \rightarrow j_{!*} C$ belongs to $i_*(\underline{D}^{t=0})$;
- (iv) M can be obtained from $j_{!*} j^* M$ via two extensions by elements of $i_* \underline{D}^{t=0}$;
- (v) $j_{!*}$ maps $\underline{Ht}_{\underline{E}}$ -monomorphisms to $\underline{Ht}_{\underline{C}}$ -ones, and $\underline{Ht}_{\underline{E}}$ -epimorphisms to $\underline{Ht}_{\underline{C}}$ -ones;
- (vi) $j_{!*} M'$ does not have non-trivial subobjects of quotient objects belonging to $i_* \underline{Ht}_{\underline{D}}$;
- (vii) the homomorphism $\underline{C}(j_{!*} M', j_{!*} Y') \rightarrow \underline{E}(M', Y')$ induced by j^* is bijective;
- (viii) if M' is simple, $j_{!*} M'$ also is;
- (ix) if M' is semisimple, then $j_{!*} M'$ can be functorially characterized as a semisimple lift of M' none of whose components are killed by j^* .

Proof. (I) We argue as in the proof of [BBD82, Theorem 1.4.10]. We consider $Y = \text{Cone}(M \rightarrow j_* B')[-1]$ and $A = \text{Cone}(Y \rightarrow i_*(t_{\underline{D}, \geq 1} i^* Y))[-1]$. We complete the commutative triangle $A \rightarrow Y \rightarrow M$ to an octahedral diagram

$$\begin{array}{ccccc}
 & j_* B' & & M & \\
 & \downarrow [1] & \searrow [1] & \nearrow [1] & \\
 & & Y & & \\
 & \downarrow [1] & \swarrow [1] & \nwarrow [1] & \\
 i_* t_{\underline{D}, \geq 1} i^* Y & & & & A
 \end{array}$$

and denote its sixth vertex by B .

Now we argue exactly as in [BBD82, Theorem 1.4.10] (using the fact that exact functors convert distinguished triangles into distinguished ones, and the ‘axioms’ of a gluing datum). We obtain $j^*(i_* t_{\underline{D}, \geq 1} i^* Y \rightarrow B \rightarrow j_* B') \cong (0 \rightarrow j^* B \rightarrow B')$; hence $j^* B \cong B'$. Next, $j^*(A \rightarrow M \rightarrow B) \cong (j^* A \rightarrow M' \rightarrow B')$; hence $j^* A \cong A'$. Furthermore, $i^*(A \rightarrow Y \rightarrow i_* t_{\underline{D}, \geq 1} i^* Y) \cong (i^* A \rightarrow i^* Y \rightarrow t_{\underline{D}, \geq 1} i^* Y)$; hence $i^* A \cong t_{\underline{D}, \leq 0} i^* Y$. It remains to note that $i^!(i_* t_{\underline{D}, \geq 1} i^* Y \rightarrow B \rightarrow j_* B') \cong (t_{\underline{D}, \geq 1} i^* Y \rightarrow i^* B \rightarrow 0)$; hence $i^! B \cong t_{\underline{D}, \geq 1} i^* Y$.

(II) (1) This is (exactly) [BBD82, Theorem 1.4.10].

(2) Obviously, if t is glued from $t_{\underline{D}}$ and $t_{\underline{E}}$, then j^* is t -exact. Since $j^* i_* = 0$, the adjunctions to i_* also yield that i_* is t -exact; see Remark 1.2.3(3).

Now, suppose that the t -exactness of i_* and j^* is fulfilled for some t -structure t' for \underline{C} . Then Remark 1.2.3(3) yields all of our t -exactness statements for t' (and so they are fulfilled for t). It follows that $1_{\underline{C}}$ is t -exact as a functor from (\underline{C}, t') to (\underline{C}, t) . Applying the other statement in Remark 1.2.3(3) we obtain $t = t'$.

(III) This is immediate from [Bon14, Proposition 1.2.3(13, 15)].

(IV) The proofs are easy applications of the results of (the end of) [BBD82, § 1.4].

(IV)(i) This is immediate from the axioms of a gluing datum and the t -exactness of j^* .

(IV)(ii) This is immediate from [BBD82, Proposition 1.4.23].

(IV)(iii) The previous assertion yields that the middle-term homology in question is killed by j^* . Since the categorical kernel of j^* is $i_* \underline{D}$, and i_* is t -exact, we obtain the result.

(IV)(iv) By assertion (IV)(i), we have an $\underline{Ht}_{\underline{C}}$ -epimorphism $a : H_0^{t_{\underline{C}}} j_! M \rightarrow j_{!*} M$, and an $\underline{Ht}_{\underline{C}}$ -monomorphism $b : j_{!*} M \rightarrow H_0^{t_{\underline{C}}} j_* M$; both of them become isomorphisms after the application

of j^* . Besides, adjunctions yield that $b \circ a$ factorizes through M . As in the proof of (IV)(iii), the result follows immediately.

(IV)(v),(vi) These are immediate from [BBD82, Corollary 1.4.25]; note that the kernel (respectively the cokernel) of the $j_{!*}$ -image of a monomorphism (respectively of an epimorphism) is necessary killed by j^* .

(IV)(vii) This is an easy consequence of (IV)(vi). Indeed, since $j^*j_{!*} \cong 1_{\underline{Ht}_E}$, it suffices to verify that the homomorphism $\underline{C}(j_{!*}M', j_{!*}Y') \rightarrow \underline{E}(M', Y')$ induced by j^* is injective. Let f be a non-zero element of $\underline{C}(j_{!*}M', j_{!*}Y')$. Then assertion (IV)(vi) yields that $\text{Im } f$ is not isomorphic to an object of $i_*(\underline{Ht}_D)$. Hence $j^* \text{Im } f \neq 0$; since j^* is t -exact we obtain $j^*f \neq 0$.

(IV)(viii) This is just [BBD82, Proposition 1.4.26].

(IV)(ix) We may assume that M' is simple. Then $j_{!*}M$ is simple also by the previous assertion. Assertion (IV)(iv) yields that $j_{!*}M$ is the only simple lift of M' . Finally, assertion (IV)(vi) implies that this characterization of $j_{!*}M$ is functorial. \square

Remark 1.5.4. So $j_{!*}M'$ is the ‘minimal’ lift of M' . As a consequence, when we ‘lift nice decompositions’ (in the proof of Theorem 3.4.1 below) it will be sufficient to check whether $j_{!*}$ ‘respects weights’. In order to verify the latter assertion, we will apply Theorem 2.5.4(II).

2. On relative motives and \mathbb{Q}_l -sheaves

In § 2.1 we introduce certain terminology for schemes and their morphisms; we also discuss our restrictions on base schemes.

In § 2.2 we recall some of basic properties of Beilinson motives over S (as defined in [CD09]).

In § 2.3 we recall certain properties of the Chow weight structure w_{Chow} for $\text{DM}_c(S)$ (as introduced in [Heb11, Bon14]); we also prove some new ‘continuity’ properties of this weight structure.

In § 2.4 we treat the étale realization of S -motives and the perverse t -structure for its target.

In § 2.5 we study weights for mixed sheaves and relate them to (the degeneration of) Chow weight spectral sequences for $H_{\mathbb{Q}_l,0}^{\text{et}}$. The latter degenerate at E_2 if S is a very reasonable scheme (we conjecture that they degenerate for a general reasonable S also). This yields that the Chow weight filtration for such (co)homology is strictly restricted by (‘motivic’) morphisms.

2.1 Schemes and morphisms: some terminology and a discussion of restrictions

All morphisms and schemes below will be separated. Besides, all schemes will be excellent Noetherian of finite Krull dimension. S will usually be our base scheme. Often $j : U \rightarrow S$ will be an open immersion, and $i : Z \rightarrow S$ will be the complementary closed embedding.

Below l will always be a prime number (as well as l'); we will usually assume l to be fixed. p will usually denote the characteristic of some scheme (so it is either a prime number or 0); usually $p \neq l$. We will say that p is the characteristic of S (only) if it is an equicharacteristic p scheme (so it is a $\text{Spec } \mathbb{F}_p$ -scheme if $p > 0$ and a $\text{Spec } \mathbb{Q}$ -scheme for $p = 0$).

Below we will identify a Zariski point (of a scheme S) with the spectrum of its residue field (sometimes we will also make no distinction between the spectrum of a field and the field itself). \mathcal{S} will denote the set of (Zariski) points of S . For $K \in \mathcal{S}$ we will denote the natural morphism $K \rightarrow S$ by j_K . We will call the dimension of the closure of K in S the dimension of K .

We now introduce some terminology for schemes and their morphisms.

DEFINITION 2.1.1. (1) We will call a scheme S *reasonable* if it is of finite type over some (Noetherian excellent separated) regular scheme S_0 of dimension less than or equal to 1.

We will only consider reasonable schemes below. For most of them one can assume S_0 to be fixed (yet we will often consider Zariski points of our schemes). In particular, when we say that a morphism of schemes is of finite type we will always assume that we have chosen a common S_0 for them.

(2) A morphism $g : X' \rightarrow Y$ will be called *essentially pro-affine* if it factorizes as $X' \xrightarrow{h} X \xrightarrow{f} Y$, where f is a finite type morphism, and h is the inverse limit of a filtered system of affine morphisms $h_i : X_i \rightarrow X$.

(3) An essentially pro-affine morphism $g' : X \rightarrow Y$ will be called *quasi-regular* if all the corresponding $f_i : X_i \rightarrow Y$ are compositions of chains of finite type smooth morphisms and finite universal homeomorphisms.

(4) A reasonable scheme S will be called *very reasonable* if there exists a surjective finite type smooth morphism $f : S' \rightarrow S$ such that S' can be presented as the inverse limit of some filtered system of schemes S'_i that are of finite type over (the spectrum of) some field K and such that all the transition morphisms $S'_i \rightarrow S'_j$ are smooth affine.

Remark 2.1.2. (I) (1) Obviously, any morphism of spectra of fields is quasi-regular, whereas any (separated) finite type scheme S over the spectrum of a field is very reasonable (since it is reasonable and one can take all $S'_i = S$).

(2) Certainly, essentially affine and quasi-regular morphisms are stable with respect to base change.

(3) Being very reasonable is an (étale) local property; this is why we did not assume that $S = S'$ in the definition above. We will use this fact in the proof of Theorem 2.5.4(II) below in order to reduce general very reasonable schemes to irreducible ones.

(II) By the celebrated theorem of Popescu (see [Pop86, Theorem 1.8] or [CD09, Theorem 4.1.5]) all *regular* morphisms of Noetherian affine schemes are quasi-regular (this result motivated our choice of the term).

(III) We have three reasons to restrict ourselves to reasonable schemes (in this paper). Yet possibly our results can be extended to arbitrary excellent Noetherian separated schemes of finite Krull dimension. We now explain this in more detail.

(1) The Chow motives over S are only known to yield the heart of a weight structure if S is of finite type over an S_0 of dimension less than or equal to 2. Yet this restriction can be avoided. In [Bon14, § 2.3] an ‘alternative’ construction of the Chow weight structure was described. It was proved that this version of w_{Chow} possesses all the properties listed in Theorem 2.3.1(II)–(IV) below, whereas (the new) parts (V)–(VI) of the theorem can be established using the explicit ‘generators’ of $(\text{DM}_c(S)_{w_{\text{Chow}} \leq 0}, \text{DM}_c(S)_{w_{\text{Chow}} \leq 0})$ given by [Bon14, Proposition 2.3.4(I2)].

(2) The existence of such an S_0 (that is regular) is also required in [CD13, Theorem 6.2.17(2)] in order to ensure the existence of a dualizing object in $\text{DM}_h(S, \mathbb{Z})$. Yet we actually only need the existence of a dualizing object in the category $D_c^b(S, \mathbb{Z}_l)$ defined in [CD13, § 7.2.18]. Possibly the latter fact can be deduced from [ILO14, Theorem XVII.0.9].

(3) Quite probably the results of [ILO14] (together with other results proved by O. Gabber) also yield the existence of a ‘reasonable’ $D_c^b \text{Sh}^{et}(S[1/l], \mathbb{Q}_l)$ together with a (self-dual) perverse t -structure for it for a not necessarily reasonable S . Yet the author has never met any claims of this sort in the literature. On the other hand, [Eke90, Theorem 6.3] says that all the arguments of [BBD82] (for l -adic étale sheaves over finite type $\text{Spec } \mathbb{F}_p$ -schemes) carry over to reasonable schemes.

(IV) We define the class of very reasonable schemes since we can prove Theorem 2.5.4(II) for them (and this is crucial for the main results of this paper). We now discuss possible generalizations of this statement.

(2) One can reduce Conjecture 2.5.3 for S to its base changes to the completions of S_0 at closed points. Next, our method of the proof of Theorem 2.5.4(II) would require ‘approximating’ the corresponding S' by schemes S'_i such that certain ‘weights’ are defined for (some version of) perverse sheaves over S_i .

Yet the author does not know of any weights of this sort in the case where S'_i are not equicharacteristic schemes (even if they are the spectra of complete discrete valuations rings); this problem seems to be related to Deligne’s weight-monodromy conjecture.

(3) Moreover, the argument we use below does not work (even) for S' being the spectrum of $K[[t]]$ (K is a field). The problem here is as follows: though any motif M over S' has a ‘model’ M_R over some $\text{Spec } R$, for R finitely generated over K (see Proposition 2.2.1(xi)), the morphism $S' \rightarrow \text{Spec } R$ does not even have to be equidimensional (since the dimension of R can be arbitrarily large). Possibly, one can apply the method used for the proof of [Ito05, Proposition 5.1], and consider the pullback $M_{R'}$ of M_R to a one-dimensional factor R' of R . Next in order to establish the degeneration of $T_{w_{\text{Chow}}(S')}(\mathcal{H}_{\mathbb{Q}_l}^{et}, M)$ one should apply the proper base change theorem (in order to relate M to $M_{R'}$) and (probably) consider a perversity for $\text{Spec } R$ that is not self-dual (so that the higher perverse inverse images with respect to morphisms $S' \rightarrow \text{Spec } R$ and $\text{Spec } R' \rightarrow \text{Spec } R$ of the terms of the Chow weight spectral sequence for perverse étale homology of M_R would vanish). One may say that such an $M_{R'}$ is a ‘clever model’ for M (a sort of Artin approximation for R).

The author may study these questions (and relate them to Rappoport–Zink spectral sequences) in a subsequent paper.

All the motives that we will consider in this paper will have rational coefficients (so that we will not mention rational coefficients in the notation; this includes Chow and DM_{gm}).

2.2 Beilinson \mathcal{S} -motives (after Cisinski and Deglise)

We list some of the properties of the triangulated categories of Beilinson motives (this is the version of relative Voevodsky’s motives with rational coefficients described by Cisinski and Deglise).

PROPOSITION 2.2.1. *Let X, Y be any (reasonable) schemes; $f : X \rightarrow Y$ is a (separated) finite type morphism.*

(i) *A tensor triangulated \mathbb{Q} -linear category $\text{DM}_c(X)$ with the unit object \mathbb{Q}_X is defined. $\text{DM}_c(X)$ is the category of constructible Beilinson motives over X , as defined (and thoroughly studied) in [CD09, § 14].*

(ii) *If S is the spectrum of a perfect field, $\text{DM}_c(S)$ is isomorphic to the category $\text{DM}_{\text{gm}} = \text{DM}_{\text{gm}}(S)$ of Voevodsky’s geometric motives (with rational coefficients) over S (see [Voe00]). Besides, $\text{DM}_{\text{gm}} = \langle \text{Chow} \rangle$ (here we consider the full embedding $\text{Chow} \rightarrow \text{DM}_{\text{gm}}$ that is a natural extension of the embedding $\text{Chow}^{\text{eff}} \rightarrow \text{DM}_{\text{gm}}^{\text{eff}}$ given by [Voe00]).*

(iii) *All $\text{DM}_c(X)$ are idempotent complete.*

(iv) *The following functors are defined: $f^* : \text{DM}_c(Y) \rightleftarrows \text{DM}_c(X) : f_*$ and $f_! : \text{DM}_c(X) \rightleftarrows \text{DM}_c(Y) : f^!$; f^* is left adjoint to f_* and $f_!$ is left adjoint to $f^!$. We call these the motivic image functors. Any of them (when f varies) yields a 2-functor from the category of reasonable schemes with separated morphisms of finite type to the 2-category of triangulated categories.*

- (v) f^* is symmetric monoidal; $f^*(\mathbb{Q}_Y) = \mathbb{Q}_X$.
- (vi) $f_* \cong f_!$ if f is proper. If f is an open immersion, we have $f^! = f^*$. More generally, $f^!(-) \cong f^*(-)(s)[2s]$ if f is smooth (everywhere) of relative dimension s .
- (vii) If X, Y are regular, and \mathcal{O}_X is a free finite-dimensional \mathcal{O}_Y -module, then the adjunction morphism $M \rightarrow f_* f^*(M)$ splits for any $M \in \text{Obj DM}_c(Y)$.
- (viii) If $i : Z \rightarrow X$ is a closed embedding, $U = X \setminus Z$, and $j : U \rightarrow X$ is the complementary open immersion, then the motivic image functors yield a gluing datum for $\text{DM}_c(-)$ (in the sense of Definition 1.5.1(1); one should set $\underline{C} = \text{DM}_c(S)$, $\underline{D} = \text{DM}_c(Z)$, and $\underline{E} = \text{DM}_c(U)$ in it).
- (ix) $\text{DM}_c(S)$ (as a triangulated category) is generated by $\{g_*(\mathbb{Q}_X)(r)\}$, where $g : X \rightarrow S$ runs through all smooth separated finite type morphisms, $r \in \mathbb{Z}$.
- (x) The functor g^* can be defined for any separated morphism g (of schemes) not necessarily of finite type; this definition respects the composition for morphisms. Moreover, one can also define $j_K^!$ for $K \in \mathcal{S}$ (see § 2.1). Besides, if for composable morphisms g, h (not necessarily of finite type) all of $h^!, g^!, (h \circ g)^!$ are defined (i.e., any of $h, g, h \circ g$ is either of finite type or of the type j_K), then $(h \circ g)^! \cong g^! \circ h^!$.
- (xi) Let $g : X' \rightarrow Y$ be an essentially pro-affine morphism (see Definition 2.1.1(2)); adopt the notation of Definition 2.1.1(2). Then $\text{DM}_c(X')$ is isomorphic to the 2-colimit of the categories $\text{DM}_c(X_i)$; in these isomorphism all the connecting functors are given by the corresponding $(-)^*$ (cf. the previous assertion).
- (xii) The functor g^* is conservative for any essentially pro-affine surjective morphism g (in particular, for a morphism of spectra of fields).
- (xiii) The family of functors j_K^* , where K runs through \mathcal{S} (see § 2.1), is conservative on $\text{DM}_c(S)$.
- (xiv) If g is a pro-finite universal homeomorphism then g^* is an equivalence of categories.
- (xv) For a Cartesian square of separated morphisms

$$\begin{array}{ccc} Y' & \xrightarrow{f'} & X' \\ \downarrow g' & & \downarrow g \\ Y & \xrightarrow{f} & X \end{array} \quad (6)$$

we have $g^* f_! \cong f_! g'^*$ (for g not necessarily of finite type) and $g'_* f'^! \cong f^! g_*$.

(xvi) Adopt the notation of the previous assertion, and assume also that g is a pro-finite universal homeomorphism. Then we also have $g^* f_* \cong f'_* g'^*$ and $g'^* f^! \cong f'^! g^*$.

(xvii) We have these isomorphisms also in the case where g is the composition of the inverse limit of smooth affine morphisms with any smooth finite type morphism.

(xviii) In the setting of assertion (viii), for any $M, N \in \text{Obj DM}(S)$ there exists a complex $\text{DM}_c(Z)(i^*(M), i^!(N)) \rightarrow \text{DM}_c(S)(M, N) \rightarrow \text{DM}_c(U)(j^* M, j^* N)$ (of abelian groups) that is exact in the middle.

Proof. Most of these statements were made in the introduction of [CD09] (and proved later in [CD09]); see [Bon14, § 1.1] for more details.

The first part of assertion (ii) is given by [CD09, Corollary 16.1.6]. The second part of it was proved in [Bon09, § 6.4].

Assertion (vii) was established in the course of proving [CD09, Theorem 14.3.3].

Assertion (xii) is just the theorem itself (cf. [CD09, Definition 2.1.7]) in the case where g is of finite type; the general case follows immediately by assertion (xi).

Assertion (xiii) easily follows from Theorem 2.3.1(IV) below.

In the case where g is finite, assertion (xiv) is given by [CD09, Proposition 2.1.9] (note that we can apply the result cited by [CD09, Theorem 14.3.3]). In order to pass to the limit in this statement one should apply assertion (xi) (once more).

(xvi) Recall that for any S the category $\mathrm{DM}_c(S)$ is a full triangulated subcategory of a certain $\mathrm{DM}(S)$; $\mathrm{DM}_c(S)$ *weakly generates* $\mathrm{DM}(S)$ (i.e., $\mathrm{DM}_c(S)^\perp = \{0\}$ in $\mathrm{DM}(S)$). Moreover, the motivic image functors can be extended to $\mathrm{DM}(-)$; g^* and its right adjoint g_* are defined for these categories for an arbitrary morphism g (of reasonable schemes). Hence in our situation g_* yields an inverse isomorphism $\mathrm{DM}_c(X') \rightarrow \mathrm{DM}_c(X)$ (since for any $M \in \mathrm{Obj} \mathrm{DM}_c(X)$ a cone of the adjunction unit morphism $M \rightarrow g_* g^* M$ is orthogonal to $\mathrm{DM}_c(X)$ by assertion (xiv)). Hence it suffices to verify $f_* g'_* \cong g_* f'_*$ and $g'_* f'^! \cong f^! g_*$. The first isomorphism is obvious, whereas the second isomorphism was established in [CD09] for g not necessarily of finite type.

Assertion (xvii) is given by [CD09, Propositions 4.3.14 and 4.3.12].

Assertion (xviii) is an easy consequence of assertion (viii). \square

2.3 The Chow weight structure for $\mathrm{DM}_c(S)$

We define $\mathrm{Chow}(S)$ as the Karoubi-closure of $\{f_*(\mathbb{Q}_X)(r)[2r]\}$ in $\mathrm{DM}_c(S)$; here $f : X \rightarrow S$ runs through all finite type projective morphisms such that X is regular, $r \in \mathbb{Z}$.

THEOREM 2.3.1. (I) *There exists a (unique) bounded weight structure w_{Chow} for $\mathrm{DM}_c(S)$ whose heart is $\mathrm{Chow}(S)$. For any $n \in \mathbb{Z}$ the functor $-(n)[2n]$ is weight-exact with respect to this weight structure.*

(II) *Let $f : X \rightarrow Y$ be a (separated) finite type morphism of schemes. Then the following statements are valid.*

- (1) $f^!$ and f_* are right weight-exact; f^* and $f_!$ are left weight-exact.
- (2) Suppose, moreover, that f is smooth. Then f^* and $f^!$ are also weight-exact.
- (3) f^* is weight-exact also if f is a finite universal homeomorphism.
- (4) If f is proper, then $f_* \mathbb{Q}_X \in \mathrm{DM}_c(Y)_{w_{\mathrm{Chow}} \leq 0}$.

(III) *Let K be a generic point of S , $M \in \mathrm{Obj} \mathrm{DM}_c(S)$.*

(1) *Suppose that $j_K^* M \in \mathrm{DM}_c(K)_{w_{\mathrm{Chow}} \geq 0}$ (respectively $j_K^* M \in \mathrm{DM}_c(K)_{w_{\mathrm{Chow}} \leq 0}$). Then there exists an open immersion $j : U \rightarrow S$, $K \in U$, such that $j^* M \in \mathrm{DM}_c(U)_{w_{\mathrm{Chow}} \geq 0}$ (respectively $j^* M \in \mathrm{DM}_c(U)_{w_{\mathrm{Chow}} \leq 0}$).*

(2) *Suppose that $j_K^* M \in \mathrm{DM}_c(K)_{w_{\mathrm{Chow}} = 0}$. Then there exists an open immersion $j : U \rightarrow S$, $K \in U$, such that $j^* M$ is a retract of $(g \circ h)_* \mathbb{Q}_P(s)[2s]$, where $h : P \rightarrow U'$ is a smooth projective morphism, U' is a regular scheme, and $g : U' \rightarrow U$ is a finite universal homeomorphism, $s \in \mathbb{Z}$.*

(IV) *$M \in \mathrm{DM}_c(S)_{w_{\mathrm{Chow}} \geq 0}$ (respectively $M \in \mathrm{DM}_c(S)_{w_{\mathrm{Chow}} \leq 0}$) if and only if for any $K \in \mathcal{S}$ we have $j_K^! (M) \in \mathrm{DM}_c(K)_{w_{\mathrm{Chow}} \geq 0}$ (respectively $j_K^* (M) \in \mathrm{DM}_c(K)_{w_{\mathrm{Chow}} \leq 0}$).*

(V) (1) f^* is left weight-exact for any morphism $f : X \rightarrow Y$ of reasonable schemes.

- (2) Consider an essentially pro-affine morphism $g : X' = \varprojlim_{i \in I} X_i \rightarrow Y$ (see Definition 2.1.1(2)) and the corresponding $g_i : X' \rightarrow X_i$. Then for an $M \in \text{Obj DM}_c(X')$ we have $M \in \text{DM}_c(X')_{w_{\text{Chow}} \leq 0}$ if and only if there exist $i \in I$ and $M_i \in \text{DM}_c(X_i)_{w_{\text{Chow}} \leq 0}$ such that $M \cong g_i^*(M_i)$.
- (3) In the notation of the previous assertion, the functor g^* is weight-exact if all the corresponding functors $f_i^* : \text{DM}_c(S)(Y) \rightarrow \text{DM}_c(S)(X_i)$ are also.
- (4) Adopt the assumptions of the previous assertion. Then for an $N \in \text{Obj DM}_c(Y)$ we have $g^*(N) \in \text{DM}_c(X')_{\geq 0}$ (respectively $g^*(N) \in \text{DM}_c(X')_{w_{\text{Chow}} \leq 0}$) if and only if for any large enough $i \in I$ we have $f_i^*(N) \in \text{DM}_c(X_i)_{w_{\text{Chow}} \geq 0}$ (respectively $f_i^*(N) \in \text{DM}_c(X_i)_{w_{\text{Chow}} \leq 0}$).
- (5) In particular, all the parts of assertion (V) can be applied if g is quasi-regular (see Definition 2.1.1(3)).

(VI) Assume that $g : X' \rightarrow Y$ is quasi-regular and surjective. Then for any $N \in \text{Obj DM}_c(Y)$, we have $g^*N \in \text{DM}_c(X')_{w_{\text{Chow}} \leq 0}$ (respectively $g^*N \in \text{DM}_c(X')_{w_{\text{Chow}} \geq 0}$) if and only if $N \in \text{DM}_c(Y)_{w_{\text{Chow}} \leq 0}$ (respectively $N \in \text{DM}_c(Y)_{w_{\text{Chow}} \geq 0}$).

Proof. Assertions (I)–(IV) were established in [Bon14]; see [Bon14, Theorems 2.1.2(I) and 2.2.1(I, II, III, VI), Lemma 2.2.4, Remark 2.3.7(4), and Proposition 2.2.3], respectively. For an f that is not quasi-projective assertion (II) was proved in [Heb11, Theorem 3.7] (where assertions (I)–(II) were proved independently and somewhat earlier than in [Bon14]); yet we will not actually need non-quasi-projective morphisms below.

(V) (1) By Proposition 1.1.3(iv) it suffices to verify that $f^*(\text{Chow}(Y)) \subset \text{DM}_c(X)_{w_{\text{Chow}} \leq 0}$. Proposition 2.2.1(xv) reduces the latter fact to assertion (II)(4).

(2) The ‘if’ part is immediate from the previous assertion.

We verify the converse assertion. Applying Proposition 1.1.3(vi) to our setting we obtain the following: there exists a finite set of non-negative indices $J \subset \mathbb{Z}$, certain projective $p^j : P^j \rightarrow X'$ (for all $j \in J$ and some regular P^j), and an $r \in \mathbb{Z}$ such that M belongs to the smallest Karoubi-closed extension-stable subclass of $\text{Obj DM}_c(X')$ containing all $p_{*}^j \mathbb{Q}_{P^j}(r)[2r-j]$. Certainly, there exists an $i \in I$ such that all p^j come via base change from certain projective $p_i^j : P_i^j \rightarrow X_i$ (for some not necessarily regular P_i^j ; see [EGA4(3), Theorems 8.8.2 and 8.10.5]). By Proposition 2.2.1(xi, xv), we can also assume that $M \cong g_i^*(M_i)$ for some M_i belonging to the smallest Karoubi-closed extension-stable subclass of $\text{Obj DM}_c(X_i)$ containing all $p_{i*}^j \mathbb{Q}_{P_i^j}(r)[2r-j]$. Finally, Proposition 1.1.3(ii) yields that $M_i \in \text{DM}_c(X_i)_{w_{\text{Chow}} \leq 0}$.

(3) We should verify the right weight-exactness of g^* . By Proposition 1.1.3(vi) it suffices to verify $M \perp g^*N$ for any $M \in \text{DM}_c(X')_{w_{\text{Chow}} \leq -1}$, $N \in \text{DM}_c(Y)_{w_{\text{Chow}} \geq 0}$. We fix M and N ; let $s \in \text{DM}_c(X')(M, g^*N)$. By the previous assertions there exist $i \in I$ and $M_i \in \text{DM}_c(X_i)_{w_{\text{Chow}} \leq -1}$ such that $M \cong g_i^*(M_i)$. Hence for the transition morphisms $g_{ji} : X_j \rightarrow X_i$ (for all $j \geq i$) we have $g_{ji}^*(M_j) \in \text{DM}_c(X_j)_{w_{\text{Chow}} \leq -1}$. Next, the weight-exactness of f_j^* yields $g_{ji}^*(M_j) \perp f_j^*(N)$. It remains to apply Proposition 2.2.1(xi).

(4) The ‘ $w_{\text{Chow}} \leq 0$ ’ part of the statement is given by assertions (V)(1)–(2). So we verify the remaining part.

Consider a weight decomposition of $N[1] : B \xrightarrow{s} N[1] \rightarrow A \rightarrow B[1]$. If $g^*N \in \text{DM}_c(Y)_{w_{\text{Chow}} \geq 0}$ then $g^*(s) = 0$ (since $\text{DM}_c(X')_{w_{\text{Chow}} \leq 0} \perp g^*(N)[1]$). Applying Proposition 2.2.1(xi) once more we obtain: for any large enough $i \in I$ we have $f_i^*(s) = 0$. Hence $f_i^*(N)$ is a direct summand of $f_i^*(A[-1])$ for all such i . Since $f_i^*(A[-1]) \in \text{DM}_c(X_i)_{w_{\text{Chow}} \geq 0}$ and $\text{DM}_c(X_i)_{w_{\text{Chow}} \geq 0}$ is Karoubi-closed in $\text{DM}_c(X_i)$, we obtain the result.

Conversely, assume that $f_i^*(N) \in \mathrm{DM}_c(X_i)_{w_{\mathrm{Chow}} \geq 0}$ for some $i \in I$. Then we obtain $f_j^*(s) = 0$. Hence $g^*(s) = 0$ also; thus g^*N is a retract of $g^*(A)[-1]$.

(5) The corresponding f_i are weight-exact by assertions (II)(2)–(3).

(VI) The ‘if’ statement is given by the previous assertion. It also yields that, in order to verify the converse implication, it suffices to do so for g being either a finite universal homeomorphism or a smooth finite type surjective morphism. The latter case is easy since g^* is an isomorphism (see Proposition 2.2.1(xiv); hence Proposition 1.1.3(v) (together with assertion (II)(3)) yields the result.

Now let g be smooth surjective. First consider the case where $Y = \mathrm{Spec} K$ (K is a field; this case is of most interest to us). Since X' is smooth (and hence Zariski-locally étale over an affine $\mathrm{Spec} K$ -space) there exists a morphism $l : \mathrm{Spec} K' \rightarrow X'$ (for a field K'/K) such that the composition (structure) morphism $k' = g \circ l : \mathrm{Spec} K' \rightarrow \mathrm{Spec} K$ is étale.

If $g^*N \in \mathrm{DM}_c(X')_{w_{\mathrm{Chow}} \leq 0}$, then $k'^*N = l^*(g^*N) \in \mathrm{DM}_c(K')_{w_{\mathrm{Chow}} \leq 0}$ (see assertion (II)(1)). Since k'_* is weight-exact, we also have $k'_*k'^*N \in \mathrm{DM}_c(K)_{w_{\mathrm{Chow}} \leq 0}$. Since N is a retract of $b'_*b'^*N$ (see Proposition 2.2.1(vii)), we conclude that $N \in \mathrm{DM}_c(K)_{w_{\mathrm{Chow}} \leq 0}$.

Next, if $g^*N \in \mathrm{DM}_c(S)_{w_{\mathrm{Chow}} \geq 0}$, then we also have $g^!N \in \mathrm{DM}_c(X)_{w_{\mathrm{Chow}} \geq 0}$ (by assertion (I) and Proposition 2.2.1(vi)). Since $k'^*N = k'^!N$, we obtain $k'^*N \in \mathrm{DM}_c(\mathrm{Spec} K')_{w_{\mathrm{Chow}} \geq 0}$. Hence $k'_*k'^*N \in \mathrm{DM}_c(K')_{w_{\mathrm{Chow}} \geq 0}$ and Proposition 2.2.1(vii) yields the result (again).

In order to reduce our claim from the case of arbitrary finite type smooth morphisms to the one when Y is the spectrum of a field, we apply assertion (IV). Here we apply the 2-functoriality of $(-)^*$ when we consider the case $g^*N \in \mathrm{DM}_c(X')_{w_{\mathrm{Chow}} \leq 0}$, and combine it with Proposition 2.2.1(vi) when $g^*N \in \mathrm{DM}_c(X')_{w_{\mathrm{Chow}} \geq 0}$. \square

Remark 2.3.2. Note also that an alternative construction of w_{Chow} over any (not necessarily reasonable) excellent separated finite-dimensional scheme S was considered in [Bon14, § 2.3]. Its functoriality properties were only studied with respect to quasi-projective morphisms; yet this is quite sufficient for our purposes.

Below we will call weight spectral sequences and weight filtrations corresponding to w_{Chow} the *Chow weight* ones.

2.4 On the étale realization of motives and the perverse t -structure

This subsection is dedicated to the justification of the following statement.

THEOREM 2.4.1. *Let $S/\mathrm{Spec} \mathbb{Z}[1/l]$ be a reasonable scheme.*

(I) (1) *We have an exact functor $\mathcal{H}_{\mathbb{Q}_l}^{\mathrm{et}}(S) : \mathrm{DM}_c(S) \rightarrow D_c^b \mathrm{Sh}^{\mathrm{et}}(S, \mathbb{Q}_l)$; the latter is the triangulated category of constructible étale \mathbb{Q}_l -sheaves over S (see below).*

(2) *The target categories $D_c^b \mathrm{Sh}^{\mathrm{et}}(-, \mathbb{Q}_l)$ of $\mathcal{H}_{\mathbb{Q}_l}^{\mathrm{et}}(-)$ are equipped with connecting functors of the type g^* for any morphism g of reasonable schemes, and also with f_* , $f_!$, and $f^!$ for a finite type morphism f .*

Moreover, $\mathcal{H}_{\mathbb{Q}_l}^{\mathrm{et}}(-)$ converts the corresponding motivic image functors (i.e., the ‘motivic’ g^ , f_* , $f_!$, and $f^!$ mentioned in Proposition 2.2.1(x),(iv)) into these étale versions.*

(II) *The category $D_c^b \mathrm{Sh}^{\mathrm{et}}(-, \mathbb{Q}_l)$ (for any reasonable $S/\mathrm{Spec} \mathbb{Z}[1/l]$) is equipped with a bounded perverse t -structure (for the self-dual perversity) that we will denote just by t ; the heart of t will be denoted by $\mathrm{Sh}_{\mathrm{per}}^{\mathrm{et}}(S, \mathbb{Q}_l)$. The collection of these t -structures over all reasonable schemes enjoys the following properties (for $f : X \rightarrow Y$ being a finite type morphism of reasonable schemes).*

- (1) If f is an immersion, then f_* and $f^!$ are left t -exact, whereas $f_!$ and f^* are right t -exact (see Definition 3.1.1(3)).
 - (2) If f is affine, then $f_!$ is left t -exact, and f_* is right t -exact.
 - (3) If f is quasi-finite affine, then f_* and $f_!$ are t -exact.
 - (4) If f is proper of relative dimension less than or equal to d , then $f_*[d](= f_![d])$ is left t -exact, and $f_*[-d]$ is right t -exact.
 - (5) If f is smooth (everywhere) of dimension d , then $f^![-d]$ and $f^*[d]$ are t -exact.
 - (6) If K is a point of S of dimension d (see § 2.1), then $j_K^*[-d]$ (respectively $j_K^![-d]$; see Remark 2.4.2 below) is left (respectively right) t -exact.
 - (7) Moreover, for $M \in \text{Obj } D_c^b Sh^{et}(S, \mathbb{Q}_l)$ we have $M \in D_c^b Sh^{et}(S, \mathbb{Q}_l)^{t \leq 0}$ (respectively $M \in D_c^b Sh^{et}(S, \mathbb{Q}_l)^{t \geq 0}$) whenever, for any $K \in \mathcal{S}$, where K is of dimension d , we have $j_K^* M[-d] \in D_c^b Sh^{et}(K, \mathbb{Q}_l)^{t \leq 0}$ (respectively $j_K^! M[-d] \in D_c^b Sh^{et}(K, \mathbb{Q}_l)^{t \leq 0}$).
 - (8) For a closed embedding $i : Z \rightarrow S$ and the complementary immersion $j : U \rightarrow S$ for $M \in \text{Obj } D_c^b Sh^{et}(S, \mathbb{Q}_l)$ we have $M \in D_c^b Sh^{et}(S, \mathbb{Q}_l)^{t \leq 0}$ (respectively $M \in D_c^b Sh^{et}(S, \mathbb{Q}_l)^{t \geq 0}$) whenever $j^* M \in D_c^b Sh^{et}(U, \mathbb{Q}_l)^{t \leq 0}$ and $i^* M \in D_c^b Sh^{et}(Z, \mathbb{Q}_l)^{t \leq 0}$ (respectively $j^* M \in D_c^b Sh^{et}(U, \mathbb{Q}_l)^{t \geq 0}$ and $i^! M \in D_c^b Sh^{et}(Z, \mathbb{Q}_l)^{t \geq 0}$).
- (III) Let $g : X \rightarrow Y$ be a surjective morphism of (reasonable) schemes. Then the following statements are valid.
- (1) g^* is conservative.
 - (2) If g is smooth (everywhere) of dimension d , then for $M \in \text{Obj } D_c^b Sh^{et}(Y, \mathbb{Q}_l)$ we have $M \in \text{Obj } D_c^b Sh^{et}(Y, \mathbb{Q}_l)^{t \leq 0}$ (respectively $M \in \text{Obj } D_c^b Sh^{et}(Y, \mathbb{Q}_l)^{t \geq 0}$) if and only if $g^*[d](M) \in \text{Obj } D_c^b Sh^{et}(X, \mathbb{Q}_l)^{t \leq 0}$ (respectively $g^*[d](M) \in \text{Obj } D_c^b Sh^{et}(X, \mathbb{Q}_l)^{t \geq 0}$).
- (IV) Let $g : X \rightarrow Y$ be a morphism such that for any irreducible component Y_i of Y the dimension of $g^{-1}(Y_i)$ equals $\dim Y_i + d$ (for some fixed $d \in \mathbb{Z}$). Then the following statements are valid.
- (1) $g^*[d] D_c^b Sh^{et}(X, \mathbb{Q}_l)^{t \leq 0} \subset D_c^b Sh^{et}(X, \mathbb{Q}_l)^{t \leq 0}$.
 - (2) Moreover, g^* is t -exact if g is a universal homeomorphism.
 - (3) Assume that g is a projective limit of smooth affine morphisms. Then $g^*[d]$ is t -exact.
- (V) For S of finite type over $\text{Spec } \mathbb{F}_p$ ($p \neq l$), $D_c^b Sh^{et}(S, \mathbb{Q}_l)$ is the category of constructible \mathbb{Q}_l -perverse sheaves as defined in [BBD82], and t is the corresponding perverse t -structure for the self-dual perversity.

Proof. (I) See [CD13, Remark 7.2.25].

(II) It is stated in [Eke90, Theorem 6.3] that the corresponding results of [BBD82] carry over to $D_c^b Sh^{et}(-, \mathbb{Q}_l)$ (over schemes that are called reasonable in this paper). The properties of $D_c^b Sh^{et}(-, \mathbb{Q}_l)$ that are required to establish these facts are given by [CD13, Theorem 6.2.17(2), Theorem 7.2.11 and Proposition 7.2.21]; also, apply Proposition 1.5.3(II2) in order to establish assertion (II)(8).

(III)(1) The statement easily follows from its $\mathbb{Z}/l\mathbb{Z}$ -coefficients analogue given by [SGA4(2), Proposition 9.1]. Note here that, the categories $D_c^b Sh^{et}(-, \mathbb{Q}_l)$ are defined as the rational hulls of certain $D_c^b Sh^{et}(-, \mathbb{Z}_l)$, whereas the latter are equipped with conservative functors to $D_c^b Sh^{et}(-, \mathbb{Z}/l\mathbb{Z})$ (see [Eke90, Theorem 6.3(i)] and [BBD82, § 2.2]). Moreover, both of these ‘change of coefficient’ functors are compatible with all the connecting functors of assertion (I)(2).

(2) This is immediate from the previous assertion together with the t -exactness of $g^*[d]$ (see assertion (I)(5)). Note here that a t -exact conservative functor cannot kill non-zero objects in the heart of the corresponding t ; hence we can apply Remark 1.2.3(4).

(IV)(1) Obvious from assertion (II)(6).

(2) We should verify the left t -exactness of g^* . Now g^* (in this case) possesses an inverse functor g_* (see [SGA4(2), Theorem 1.1]; note that it is sufficient to know this statement for complexes of $\mathbb{Z}/l\mathbb{Z}$ -module sheaves). Hence it suffices to note that the functors $-_*$ possess the base change property with respect to $-^!$ (by [SGA4(3), Corollary 3.1.12.3]): for any cartesian diagram

$$\begin{array}{ccc} x & \xrightarrow{i_X} & X \\ \downarrow g' & & \downarrow g \\ y & \xrightarrow{i_Y} & Y \end{array} \quad (7)$$

we have $g'_* \circ i_X^! \cong i_Y^! \circ g_*$ (note again that it suffices to verify the latter fact for the categories $D_c^b Sh^{et}(-, \mathbb{Z}/l\mathbb{Z})$).

(3) Again, we should verify the left t -exactness property. Let y be a Zariski point of Y ; consider diagram (7) again. Then we have $g'^* \circ i_Y^! \cong i_X^! \circ g^*$; this statement can be proved via reducing to the case of $\mathbb{Z}/l\mathbb{Z}$ -coefficients and applying the argument used in the proof of [CD09, Proposition 4.3.12]. This reduces the statement to the case where Y is (the spectrum of) a field. In the latter case it is an easy consequence of absolute purity (see [ILO14, Theorem XVI.3.1.1]).

(V) This is just a partial case of the definition of $D_c^b Sh^{et}(-, \mathbb{Q}_l)$. \square

Remark 2.4.2. (1) For a $K \in \mathcal{S}$ (see §2.1) one can define the étale version of $j_K^!$ using the ‘classical’ method (see [BBD82, §2.2.12]): for its decomposition $K \xrightarrow{i_K} \overline{K} \xrightarrow{j_{\overline{K}}} S$ (\overline{K} is the closure of K in S) one should take $j_K^! = i_K^* \circ j_{\overline{K}}^!$. This definition is certainly compatible with its motivic version (see [Bon14, Remark 1.1.3(2)]).

(2) Note that we will not really need parts (II)(2)–(3) of our theorem below, and we will only need part (II)(4) in the (trivial) case of finite extensions of fields.

2.5 On weights for perverse S -sheaves; the degeneration of Chow weight spectral sequences for étale homology

Below we will need certain ‘weights’ for $\mathcal{H}_{\mathbb{Q}_l}^{et}(-)$. First we recall that in certain cases weights are defined on $D_c^b Sh^{et}(S, \mathbb{Q}_l)$.

PROPOSITION 2.5.1. (I) *Let S be a finite type (separated) $\text{Spec } \mathbb{F}_p$ -scheme (for a prime $p \neq l$). Then there exist certain $D_c^b Sh^{et}(S, \mathbb{Q}_l)_{w \leq 0}, D_c^b Sh^{et}(S, \mathbb{Q}_l)_{w \geq 0} \subset \text{Obj } D_c^b Sh^{et}(S, \mathbb{Q}_l)$, that satisfy the following properties.*

(i) *For any $m \in \mathbb{Z}$ denote $D_c^b Sh^{et}(S, \mathbb{Q}_l)_{w \leq 0}[m]$ by $D_c^b Sh^{et}(S, \mathbb{Q}_l)_{w \leq m}$ and denote $D_c^b Sh^{et}(S, \mathbb{Q}_l)_{w \geq 0}[m]$ by $D_c^b Sh^{et}(S, \mathbb{Q}_l)_{w \geq m}$. Then for $X \in \text{Obj } D_c^b Sh^{et}(S, \mathbb{Q}_l)$ we have $X \in D_c^b Sh^{et}(S, \mathbb{Q}_l)_{w \leq m}$ (respectively $X \in D_c^b Sh^{et}(S, \mathbb{Q}_l)_{w \geq m}$) if and only if for any $j \in \mathbb{Z}$ we have $H_j^t(X) \in D_c^b Sh^{et}(S, \mathbb{Q}_l)_{w \leq m+j}$ (respectively $H_j^t(X) \in D_c^b Sh^{et}(S, \mathbb{Q}_l)_{w \geq m+j}$).*

(ii) *Denote $D_c^b Sh^{et}(S, \mathbb{Q}_l)_{w \leq m} \cap D_c^b Sh^{et}(S, \mathbb{Q}_l)^{t=0}$ by $Sh_{\text{per}}^{et}(S, \mathbb{Q}_l)_{w \leq m}$, and denote $D_c^b Sh^{et}(S, \mathbb{Q}_l)_{w \geq m} \cap D_c^b Sh^{et}(S, \mathbb{Q}_l)^{t=0}$ by $Sh_{\text{per}}^{et}(S, \mathbb{Q}_l)_{w \geq m}$; $Sh_{\text{per}}^{et}(S, \mathbb{Q}_l)_{w=m} = Sh_{\text{per}}^{et}(S, \mathbb{Q}_l)_{w \leq m} \cap Sh_{\text{per}}^{et}(S, \mathbb{Q}_l)_{w \geq m}$. Then $Sh_{\text{per}}^{et}(S, \mathbb{Q}_l)_{w=m}$ yield exact abelian subcategories of $Sh_{\text{per}}^{et}(S, \mathbb{Q}_l)$ that contain all $Sh_{\text{per}}^{et}(S, \mathbb{Q}_l)$ -subquotients of their objects. Besides, for $j \neq m \in \mathbb{Z}$ we have $Sh_{\text{per}}^{et}(S, \mathbb{Q}_l)_{w=m} \perp Sh_{\text{per}}^{et}(S, \mathbb{Q}_l)_{w=j}$.*

(iii) For any open immersion $j : U \rightarrow S$ and $m \in \mathbb{Z}$ (the perverse sheaf version of) the functor $j_{!*}$ sends $Sh_{\text{per}}^{et}(S, \mathbb{Q}_l)(U)_{w \leq m}$ into $Sh_{\text{per}}^{et}(S, \mathbb{Q}_l)_{w \leq m}$, and sends $Sh_{\text{per}}^{et}(S, \mathbb{Q}_l)(U)_{w \geq m}$ into $Sh_{\text{per}}^{et}(S, \mathbb{Q}_l)_{w \geq m}$.

(iv) $\mathcal{H}_{\mathbb{Q}_l}^{et}$ is ‘weight-exact’, i.e., it sends $DM_c(S)_{w_{\text{Chow}} \leq m}$ into $D_c^b Sh^{et}(S, \mathbb{Q}_l)_{w \leq m}$, and sends $DM_c(S)_{w_{\text{Chow}} \geq m}$ into $D_c^b Sh^{et}(S, \mathbb{Q}_l)_{w \geq m}$.

(II) Let S be a finite type (separated) $\text{Spec } \mathbb{Q}$ -scheme. Present S as an inverse limit of finite type $\text{Spec } \mathbb{Z}[1/l]$ -schemes S_i (with connecting morphisms being open embeddings), and define $\mathcal{H}_{\mathbb{Q}_l}^{et}(S)$ as the direct limit of $\mathcal{H}_{\mathbb{Q}_l}^{et}(S_i)$ (its target $\tilde{D}_c^b Sh^{et}(S, \mathbb{Q}_l)$ is the 2-colimit of the corresponding $D_c^b Sh^{et}(S_i, \mathbb{Q}_l)$). Then $\mathcal{H}_{\mathbb{Q}_l}^{et}(S)$ can be factorized through $\tilde{\mathcal{H}}_{\mathbb{Q}_l}^{et}(S)$. Moreover, $\tilde{D}_c^b Sh^{et}(S, \mathbb{Q}_l)$ possesses a (perverse) t -structure that is compatible with t (with respect to this connecting functor). Lastly, for $\tilde{D}_c^b Sh^{et}(S, \mathbb{Q}_l)$ one can define weights such that the analogues of all of the assertions of part (I) are fulfilled.

Proof. (I) All the assertions except (iv) are well-known properties of weights of mixed complexes of sheaves that were established in [BBD82, § 5], whereas assertion (iv) was verified in [Bon14, § 3.6].

(II) The t -exactness of the connection functor $\tilde{D}_c^b Sh^{et}(S, \mathbb{Q}_l) \rightarrow D_c^b Sh^{et}(S, \mathbb{Q}_l)$ is immediate from Theorem 2.4.1(IV)(3). Everything else was verified in [Hub97, § 3], except the analogue of assertion (I)(iv) that was established in [Bon14, § 3.4] (in this case). \square

Remark 2.5.2. (1) For S being of finite type over $\text{Spec } \mathbb{F}_p$ there exists a weight filtration (in the sense of Definition 1.3.4) for the heart of the perverse t -structure for the category $D_m^b(X, \mathbb{Q}_l)$ of mixed complexes of sheaves; see [BBD82, Theorem 5.3.5].

On the other hand, the corresponding ‘pure factors’ \underline{A}_m are not semisimple. Indeed, there are non-trivial 1-extensions of pure (perverse) sheaves even in the case where $S = \text{Spec } \mathbb{F}_p$ (since for ‘abstract’ pure Galois representations of $\text{Gal}(\mathbb{F}_p)$ the action of the Frobenius does not have to be semisimple due to the fact that one cannot impose any polarizability restrictions on these representations). Since all \underline{C} -extensions in the heart of a weight structure for \underline{C} necessarily split (immediately from the orthogonality axiom), this weight filtration does not yield a weight structure for this category.

The author made an attempt to ‘axiomatize’ this setting by introducing the notion of a *relative weight structure*; see [Bon14, §§ 3.5–3.6].

(2) For S being of finite type over $\text{Spec } \mathbb{Q}$ the category of mixed perverse sheaves $\tilde{D}_c^b Sh^{et}(S, \mathbb{Q}_l)$ does not possess a weight filtration (in our sense) at all; cf. the warning preceding [Hub97, Proposition 3.4]. The problem here is that (due to the non-vanishing of the corresponding extension groups) one can construct a mixed perverse sheaf whose pure factors are ‘in the wrong order’. Besides, note that $\tilde{D}_c^b Sh^{et}(S, \mathbb{Q}_l)$ is not isomorphic to $D_c^b Sh^{et}(S, \mathbb{Q}_l)$; see the end of [Hub97, § 1].

(3) So, in both of these cases we do not have (a ‘true’) weight structure for (any sort of) triangulated category of complexes of mixed S -sheaves; hence the properties of (relative) motives are somewhat better than those of mixed complexes of sheaves (even over a finitely generated base).

We now prove the main properties of Chow weight spectral sequences for $H_{\mathbb{Q}_l, 0}^{et}$. To this end we state the following conjecture.

CONJECTURE 2.5.3. *The spectral sequence $T_{w_{\text{Chow}}}(H_{\mathbb{Q}_l, 0}^{et}, M)$ degenerates at E_2 for any $M \in \text{Obj } DM_c(S)$ (for any reasonable S).*

THEOREM 2.5.4. (I) (1) Let S be a finite type separated scheme over $\mathrm{Spec} \mathbb{F}_p$, $M \in \mathrm{Obj} \mathrm{DM}_c(S)$, $H = H_{\mathbb{Q}_l,0}^{et}$. Then $E_s^{pq} T_{w_{\mathrm{Chow}}}(H, M) \in Sh_{\mathrm{per}}^{et}(S, \mathbb{Q}_l)_{w=q}$ for any $p, q \in \mathbb{Z}$, $s > 0$. Besides, $(W_m H)(M)$ is a filtration of $H(M)$ whose m th factor belongs to $Sh_{\mathrm{per}}^{et}(S, \mathbb{Q}_l)_{w=m}$ (for all $m \in \mathbb{Z}$). Moreover, this filtration is uniquely and functorially characterized by the latter property.

(2) For S being separated of finite type over $\mathrm{Spec} \mathbb{Q}$ and $H = \tilde{H}_{\mathbb{Q}_l,0}^{et}$ the (obvious) analogue of assertion (I)(1) is fulfilled.

(II) Let S be a very reasonable scheme of characteristic p ($p \neq l$; it can be 0). Then the following statements are valid.

(1) Conjecture 2.5.3 holds.

(2) Let $j : U \rightarrow S$ be an open embedding; denote the complementary closed embedding by i . For $M \in \mathrm{DM}_c(U)_{w_{\mathrm{Chow}} \geq s}$ (respectively $M \in \mathrm{DM}_c(U)_{w_{\mathrm{Chow}} \leq s}$) suppose that $H_{\mathbb{Q}_l,m}^{et}(M) = 0$ for all $m \neq 0$.

Then $(W_{s+m+1} H_{\mathbb{Q}_l,m}^{et})(i^! j_! M) = 0$ for any $m > 0$ (respectively $(W_{s+m} H_{\mathbb{Q}_l,m}^{et})(i^* j_* M) = H_{\mathbb{Q}_l,m}^{et}(i^* j_* M)$ for any $m < 0$).

Proof. (I) (1) Proposition 2.5.1(I)(iv) yields that $E_1^{pq} T_{w_{\mathrm{Chow}}}(H_{\mathbb{Q}_l,0}^{et}, M) \in Sh_{\mathrm{per}}^{et}(S, \mathbb{Q}_l)_{w=q}$. Hence the same is true for $E_s^{pq}(T)$ for any $s \geq 1$ (since E_s^{pq} is a subfactor of $E_1^{pq}(T)$; here we apply Proposition 2.5.1(I)(ii)). Hence $E_{\infty}^{p+q}(T) \in Sh_{\mathrm{per}}^{et}(S, \mathbb{Q}_l)_{w=q}$ also, and we obtain that the factors of the Chow weight filtration are of the weights prescribed. Now, the orthogonality of (subquotients of) perverse sheaves of distinct weight yields that this condition determines the filtration in a functorial way.

The same argument proves assertion (I)(2).

(II) (1) We verify the degeneration of $T_{w_{\mathrm{Chow}}}(H_{\mathbb{Q}_l,0}^{et}, M)$ for some fixed $M \in \mathrm{Obj} \mathrm{DM}_c(S)$ via reducing it to the case where S is of finite type over the corresponding prime field (in three steps). In each of these reduction steps we will apply Remark 1.3.3(3).

Let $f : S' \rightarrow S$ be a smooth surjective morphism as in Definition 2.1.1(4). Note that we can assume that f is equidimensional and that all the connected components of S' are irreducible. Now it suffices to verify the degeneration for $T_{w_{\mathrm{Chow}}(S')}(H_{\mathbb{Q}_l,0}^{et}(S'), f^* M)$ instead of $T_{w_{\mathrm{Chow}}(S)}(H_{\mathbb{Q}_l,0}^{et}(S), M)$. Indeed, the étale f^* is t -exact (up to a shift) and conservative (see Theorem 2.4.1(II)(5), (III)(1)), whereas the motivic f^* is w_{Chow} -weight exact (see Theorem 2.3.1(II)(2)); hence we can construct a diagram of type (4) and apply Remark 1.3.3(3) (for the first time). Moreover, it suffices to verify the degeneration of $T_{w_{\mathrm{Chow}}}$ for the restrictions of $f^* M$ to all of the connected components of S' . For this reason we will assume below that S' is irreducible.

Next for $S' = \varinjlim S'_i$ we apply Proposition 2.2.1(xi) in order to find an index i such that $f^* M = g^* M_i$ for the corresponding $g : S' \rightarrow S'_i$ and some $M_i \in \mathrm{Obj} \mathrm{DM}_c(S'_i)$. Since the étale g^* is t -exact up to a shift (see Theorem 2.4.1(IV)(3)) and the motivic one is weight-exact (see Theorem 2.3.1(V)(5)), we reduce the statement to the degeneration of $T_{w_{\mathrm{Chow}}(S'_i)}(H_{\mathbb{Q}_l,0}^{et}(S'_i), M_i)$.

Next, since S'_i is of finite type over a field, there exists a finite extension of $\mathrm{Spec} \mathbb{F}_p$ or $\mathrm{Spec} \mathbb{Q}$ (in the case $p = 0$) such that S'_i is defined over it. Hence (by Proposition 2.2.1(xi)) there exists a morphism $h : S'_i \rightarrow S''$ that satisfies the following properties: S'' is of finite type over the corresponding prime field, $M' = h^* M''$ for some $M'' \in \mathrm{Obj} \mathrm{DM}_c(S'')$, and h is the composition of a pro-finite universal homeomorphism with a projective limit of smooth affine morphisms. Hence (by the same arguments as above; see also Theorem 2.4.1(IV)(2)) it suffices to verify the statement for $T_{w_{\mathrm{Chow}}(S'')}(H_{\mathbb{Q}_l,0}^{et}(S''), M'')$.

Now consider the case $p > 0$ (i.e., S' is of finite type over $\text{Spec } \mathbb{F}_p$). In this case assertion (I)(1) yields that $E_s^{pq}(T_{w_{\text{Chow}}(S'')}(H_{\mathbb{Q}_l,0}^{et}(S''), M'')) \in Sh_{\text{per}}^{et}(S'')_{w=q}$ (for any $p, q \in \mathbb{Z}$, $s > 0$). Now (by Proposition 2.5.1(I)(ii)) there are no non-zero morphisms between distinct $Sh_{\text{per}}^{et}(S'')_{w=m}$. Hence all the connecting morphisms for $E_s(T)$ vanish for all $s > 1$, and we obtain the result.

In the case where S is of finite type over $\text{Spec } \mathbb{Q}$ we note that the same argument proves the degeneration of $T_{w_{\text{Chow}}(S'')}(H_{\mathbb{Q}_l,0}^{et}(S''), M'')$; hence the functoriality of Chow weight spectral sequences (with respect to H ; see Proposition 1.3.2(I)) yields the assertion desired.

(2) The same reduction arguments as above (along with Proposition 2.2.1(xv)–(xvii)) enable us to assume that S is of finite type over $\text{Spec } \mathbb{F}_p$ (for $p \neq 0$) or over $\text{Spec } \mathbb{Q}$. In this case Theorem 2.4.1(I,II) along with assertion (I) allow us to translate our assertion into the corresponding analogue for weights on $D_c^b Sh^{et}(S, \mathbb{Q}_l)$.

Now, for $S/\text{Spec } \mathbb{F}_p$ we have $i^! D_c^b Sh^{et}(S, \mathbb{Q}_l)_{w \leq s} \subset D_c^b Sh^{et}(Z, \mathbb{Q}_l)_{w \leq s}$ and $i^* D_c^b Sh^{et}(S, \mathbb{Q}_l)_{w \geq s} \subset D_c^b Sh^{et}(Z, \mathbb{Q}_l)_{w \geq s}$ by [BBD82, §5.1.14, (i*) and (i)]. By Proposition 1.5.3(IV)(ii) it suffices to note that j_{1*} respects weights of mixed sheaves; this is [BBD82, Corollary 5.3.2].

For $S/\text{Spec } \mathbb{Q}$ it suffices to verify the assertion for $\tilde{H}_{\mathbb{Q}_l,m}^{et}$ instead of $H_{\mathbb{Q}_l,m}^{et}$. In this setting we can apply the remark after [Hub97, Definition 3.3 and Corollary 3.5] (instead of the results of [BBD82] cited above). \square

Remark 2.5.5. (1) Using Verdier duality (for motives or sheaves), one can easily carry over the results above from étale homology to étale cohomology.

(2) In the characteristic 0 case of Theorem 2.5.4(II), we could have tried to use M. Saito's Hodge modules in our weight arguments (in order to avoid the usage of $\tilde{\mathcal{H}}_{\mathbb{Q}_l}^{et}$). The main problem here is that (to the knowledge of the author) no ‘Hodge module realization’ of motives is known to exist at the moment (however, see the proof of [Wil12, Proposition 7.6] for reasoning that avoids this difficulty).

Alternatively, one could try to reduce the characteristic 0 case to the positive characteristic one using the methods and results of [BBD82, §6].

3. On the existence of a (nice) motivic t -structure

In §3.1 we define a (motivic) t -structure t_l for $\text{DM}_c(S)$ as the one that is strictly compatible with the perverse t -structure for the \mathbb{Q}_l -étale homology (cf. [Bei02, §2.10]). We also study the functoriality of this definition.

In §3.2 we reduce the existence of t_l to the case where S is the (spectrum of) a universal domain (of characteristic distinct from l). Moreover, the existence of t_l over universal domains automatically yields that Chow weight filtrations and Chow weight spectral sequences can be lifted from $Sh_{\text{per}}^{et}(S, \mathbb{Q}_l)$ to motives. When S is a very reasonable scheme, the weight filtration for $\underline{H}t_l$ obtained this way is strictly compatible with morphisms.

In §3.3 we study certain properties of motives that follow from the *niceness* of t_l (i.e., from its transversality with w_{Chow}).

In §3.4 we apply these results (in a Noetherian induction step). We prove that a nice t_l exists over an arbitrary very reasonable scheme S of characteristic p if such a t_l exists over some universal domain of the same characteristic.

3.1 The motivic t -structure (for $S/\text{Spec } \mathbb{Z}[1/l]$)

Till §4.3 we will fix some prime l , and will usually assume that all the schemes we consider are $\text{Spec } \mathbb{Z}[1/l]$ -schemes. In this case we will define the motivic t -structure in terms of $\mathcal{H}_{\mathbb{Q}_l}^{et}$; we will treat the question whether it actually depends on l later.

DEFINITION 3.1.1. Let S be a (reasonable) scheme.

(1) Consider the class $\mathrm{DM}_c(S)^{t_i \leq 0}$ (respectively $\mathrm{DM}_c(S)^{t_i \geq 0}$) consisting of those $M \in \mathrm{Obj} \mathrm{DM}_c(S)$ that satisfy: $\mathcal{H}_{\mathbb{Q}_l}^{et}(M) \in D_c^b Sh^{et}(S[1/l], \mathbb{Q}_l)^{t \leq 0}$ (respectively $\mathcal{H}_{\mathbb{Q}_l}^{et}(M) \in D_c^b Sh^{et}(S[1/l], \mathbb{Q}_l)^{t \geq 0}$; see Theorem 2.4.1).

(2) For a $\mathrm{Spec} \mathbb{Z}[1/l]$ -scheme S if $(\mathrm{DM}_c(S)^{t_i \leq 0}, \mathrm{DM}_c(S)^{t_i \geq 0})$ yield a t -structure for $\mathrm{DM}_c(S)$, we will say that (the t -structure) t_l exists for $\mathrm{DM}_c(S)$, or that it exists over S . We will denote the heart of t_l (in this case) by $\mathrm{MM}(S)$.

(3) We will use the term ‘(left, right, or both) t -exact functor’ for functors between certain $\mathrm{DM}_c(-)$ that respect (the ‘halves of’) t_l in the corresponding way without (necessarily) assuming that t_l yields a t -structure.

(4) If t_l exists for $\mathrm{DM}_c(S)$, we will say that it is *nice* if it is transversal to w_{Chow} .

Remark 3.1.2. If t_l exists over S , then it is automatically bounded, since the étale homology of any object of $\mathrm{DM}_c(S)$ is. The latter fact is immediate from Proposition 2.2.1(ix) and Theorem 2.4.1(I, II).

In particular, we obtain that t_l is non-degenerate (see Remark 1.2.3(4)) and that $\mathcal{H}_{\mathbb{Q}_l}^{et}$ is conservative. Note that the existence of t_l is a very strong assumption!

We will need some functoriality properties of $(\mathrm{DM}_c(-)^{t_i \leq 0}, \mathrm{DM}_c(-)^{t_i \geq 0})$ below; certainly, they become even more interesting (in their own right) if t_l exists.

LEMMA 3.1.3. Let $f : X \rightarrow Y$ be a morphism of schemes. Then the following statements are valid.

(1) If f is an immersion, then f_* and $f^!$ are left t -exact, whereas $f_!$ and f^* are right t -exact (see Definition 3.1.1(3)).

(2) If f is affine, then $f_!$ is left t -exact, and f_* is right t -exact.

(3) If f is quasi-finite affine, then f_* and $f_!$ are t -exact.

(4) If f is proper of relative dimension less than or equal to d , then $f_*[d](=f_![d])$ is left t -exact, and $f_*[-d]$ is right t -exact.

(5) If f is smooth of dimension d , then $f^![-d]$ and $f^*[d]$ are t -exact.

(6) If K is a point of S of dimension d (see § 2.1), then $j_K^*[-d]$ (respectively $j_K^![-d]$) is left (respectively right) t -exact.

(7) Moreover, for $M \in \mathrm{Obj} \mathrm{DM}_c(S)$ we have $M \in \mathrm{DM}_c(S)^{t_i \leq 0}$ (respectively $M \in \mathrm{DM}_c(S)^{t_i \geq 0}$) if and only if for any $K \in \mathcal{S}$, where K is of dimension d , we have $j_K^* M[-d] \in \mathrm{DM}_c(K)^{t_i \leq 0}$ (respectively $j_K^! M[-d] \in \mathrm{DM}_c(K)^{t_i \geq 0}$).

(8) For a closed embedding $i : Z \rightarrow S$ and the complementary immersion $j : U \rightarrow S$ for $M \in \mathrm{Obj} \mathrm{DM}_c(S)$ we have $M \in \mathrm{DM}_c(S)^{t_i \leq 0}$ (respectively $M \in \mathrm{DM}_c(S)^{t_i \geq 0}$) if and only if $j^* M \in \mathrm{DM}_c(U)^{t_i \leq 0}$ and $i^* M \in \mathrm{DM}_c(Z)^{t_i \leq 0}$ (respectively $j^* M \in \mathrm{DM}_c(U)^{t_i \geq 0}$ and $i^! M \in \mathrm{DM}_c(Z)^{t_i \geq 0}$).

Proof. Immediate from Theorem 2.4.1(II). \square

We now formulate the first of the main results of this paper.

THEOREM 3.1.4. Suppose that for any point K of (a reasonable $\mathrm{Spec} \mathbb{Z}[1/l]$ -)scheme S there exists t_l for $\mathrm{DM}_c(K)$. Then t_l exists for $\mathrm{DM}_c(S)$ also.

3.2 The proof of the ‘globalization’ theorem for t_l

Till § 4.3 we will assume that S is a (reasonable) $\mathrm{Spec} \mathbb{Z}[1/l]$ -scheme.

We will need the following statement.

LEMMA 3.2.1. Let K be a generic point of a scheme U' whose dimension is d (see § 2.1); denote the morphism $K \rightarrow U'$ by j_K .

Let M be an object of $\mathrm{DM}_c(U')$ and assume that $j_K^* M[-d] \in \mathrm{DM}_c(K)^{t_l \leq 0}$ (respectively $j_K^* M[-d] \in \mathrm{DM}_c(K)^{t_l \geq 0}$). Then there exists an open immersion $j : U'' \rightarrow U'$, $K \in U''$, such that $j^* M \in \mathrm{DM}_c(U'')^{t_l \leq 0}$ (respectively $j^* M \in \mathrm{DM}_c(U'')^{t_l \geq 0}$).

Proof. Theorem 2.4.1(I,II) reduces this fact to its $D_c^b Sh^{et}(S, \mathbb{Q}_l)$ -version. Applying Verdier duality, we obtain the following: for any $C \in \mathrm{Obj} D_c^b Sh^{et}(U', \mathbb{Q}_l)$, if $j_K^*(C)[-d] \in D_c^b Sh^{et}(K, \mathbb{Q}_l)^{t_l \leq 0}$ then there exists an open immersion $j : U'' \rightarrow U'$, $K \in U'$, such that $j^* C \in D_c^b Sh^{et}(U'', \mathbb{Q}_l)^{t_l \leq 0}$. Now, over K the perverse t -structure for $D_c^b Sh^{et}(K, \mathbb{Q}_l)$ coincides with the ‘canonical’ one (corresponding to the canonical t -structure for the derived category $D_c^b Sh^{et}(K, \mathbb{Q}_l)$), whereas over any U'' any (‘ordinary’) constructible \mathbb{Q}_l -sheaf belongs to $D_c^b Sh^{et}(U'', \mathbb{Q}_l)^{t_l \leq 0}$. Considering the canonical homology of C (note that j^* and j_K^* are exact when restricted to the category of ‘ordinary’ \mathbb{Q}_l -sheaves) we obtain that it suffices to verify that if the stalk of some constructible \mathbb{Q}_l -sheaf T at K is zero, then for some open $U'' \subset U'$, $K \in U''$, we have $j^* T = 0$. This is immediate from [FK88, Proposition I.12.10]. Note that one can apply the method of the proof of [FK88, Proposition I.12.10] in our (more general) setting by [Eke90, Theorem 6.3(i)]. \square

We now prove Theorem 3.1.4.

We should prove that $(\mathrm{DM}_c(S)^{t_l \leq 0}, \mathrm{DM}_c(S)^{t_l \geq 0})$ (see Definition 3.1.1(1)) yield a t -structure for $\mathrm{DM}_c(S)$. Obviously, to this end it suffices to verify that for $\mathrm{DM}_c(S)^{t_l \leq 0}$ and $\mathrm{DM}_c(S)^{t_l \geq 0}$ prescribed by Definition 3.1.1 we have the orthogonality property, and that t_l -decompositions exist.

The proof of orthogonality uses an argument contained in the proof of [Bon14, Proposition 2.2.3]. We apply Noetherian induction. Suppose that the assertion is fulfilled over any (proper) closed subscheme of S .

For any (fixed) $M \in \mathrm{DM}_c(S)^{t_l \leq 0}$, $N \in \mathrm{DM}_c(S)^{t_l \geq 1}$, $h \in \mathrm{DM}_c(S)(M, N)$, we should prove that $h = 0$.

Let K be a generic point of S of dimension d . Lemma 3.1.3(6) yields that $j_K^* M[-d] \in \mathrm{DM}_c(K)^{t_l \leq 0}$, $j_K^! N[-d] \in \mathrm{DM}_c(K)^{t_l \geq 1}$. Hence $j_K^* h = 0$ (since t_l exists for K -motives). Hence (by Proposition 2.2.1(xi)) there exists an open immersion $j : U \rightarrow S$, $K \in U$, such that $j^* h = 0$. Let $i : Z \rightarrow S$ denote the complementary closed embedding; Lemma 3.1.3(1) yields that $i^*(M) \in \mathrm{DM}_c(Z)^{t_l \leq 0}$, $i^! N \in \mathrm{DM}_c(Z)^{t_l \geq 1}$. By the inductive assumption (applied to Z) we have $\mathrm{DM}_c(Z)(i^*(M), i^! N) = \{0\}$. Hence Proposition 2.2.1(xviii) yields the assertion.

It remains to verify the existence of a t_l -decomposition for an $M \in \mathrm{Obj} \mathrm{DM}_c(S)$. We use the method of the proof similar to that of [Bon14, Proposition 2.3.4]. Again, we apply Noetherian induction and assume that the assertion is fulfilled over any proper closed subscheme of S .

We choose some generic point K of S . We consider the t_l -decomposition

$$A_K[-d] \rightarrow j_K^* M[-d] \rightarrow B_K[-d] \quad (8)$$

(of $j_K^* M[-d]$ in $\mathrm{DM}_c(K)$). We verify that there exists an open immersion $j : U \rightarrow S$ containing S such that (8) (shifted by $[d]$) lifts to a t_l -decomposition of $j^* M$. By Proposition 2.2.1(xi) it suffices to verify the following: for any open $U' \subset S$ containing K and any $A_{U'}, B_{U'} \in \mathrm{DM}_c(U')$ such that the ‘restriction’ of $(A_{U'}, B_{U'})$ to K equals (A_K, B_K) , there exists an open $U \subset U'$ (containing K) such that ‘restrictions’ A_U, B_U of $A_{U'}, B_{U'}$ to U belong to $\mathrm{DM}_c(U)^{t_l \leq 0}$ and to $\mathrm{DM}_c(U)^{t_l \geq 1}$, respectively. This is immediate from Lemma 3.2.1.

Again, we consider the closed embedding $i : Z \rightarrow S$ complementary to j . Now the idea is that t_l for $\mathrm{DM}_c(S)$ can be glued from those for $\mathrm{DM}_c(U)$ and $\mathrm{DM}_c(Z)$. Though we only have t_l -decompositions in the latter category (by the inductive assumption), this is sufficient to

construct the t_l -decomposition of M . Indeed, by Proposition 1.5.3(I) there exists a distinguished triangle $A \rightarrow M \rightarrow B$ such that $j^*A \in \mathrm{DM}_c(U)^{t_l \leq 0}$ and $i^*A \in \mathrm{DM}_c(Z)^{t_l \leq 0}$ (respectively $j^*B \in \mathrm{DM}_c(U)^{t_l \geq 1}$ and $i^!B \in \mathrm{DM}_c(Z)^{t_l \geq 1}$). By Lemma 3.1.3(7), this triangle yields the t_l -decomposition of M .

Remark 3.2.2. (1) Actually, we do not need a complete characterization of t_l for the proof. We only need a pointwise characterization of t_l (cf. Lemma 3.1.3(7)) and Lemma 3.2.1 for it.

We also make the following observation: if we have any t -structures for $\mathrm{DM}_c(S)$, $\mathrm{DM}_c(K)$, and $\mathrm{DM}_c(U)$ for any U such that all possible j^* and $j_K^*[-d]$ are t -exact, then the statement of Lemma 3.2.1 for these t -structures is fulfilled automatically. Indeed, Proposition 2.2.1(xi) implies that $j^*(M^{t \geq 1})$ (respectively $j^*(M^{t \leq -1})$) vanishes for some U , since $j_K^*(M^{t \geq 1})$ (respectively $j_K^*(M^{t \leq -1})$) does.

Still, the author does not know how to verify Lemma 3.2.1 for the version of (the description of) the motivic t -structure (over fields) given by [Bei02, Proposition 4.5].

(2) Lemma 3.1.3(7) yields that t_l does not depend on the choice of (a version of) $\mathcal{H}_{\mathbb{Q}_l}^{et}$ over S ; see also Remark 4.1.2 below.

We now prove that it suffices to verify the conservativity of $\mathcal{H}_{\mathbb{Q}_l}^{et}$ and the existence of t_l over universal domains.

PROPOSITION 3.2.3. (1) Assume that $\mathcal{H}_{\mathbb{Q}_l}^{et}$ is conservative on $\mathrm{DM}_c(K)$ for all $K \in \mathcal{S}$. Then the same is true for $\mathrm{DM}_c(S)$.

(2) Suppose that $\mathcal{H}_{\mathbb{Q}_l}^{et}$ is conservative over some set of universal domains K_i of certain characteristics $p_i \neq l$ (one of the p_i can be 0). Then $\mathcal{H}_{\mathbb{Q}_l}^{et}$ is conservative over any (reasonable) S that satisfies the following conditions: the characteristic of any point of S is one of the p_i .

(3) Suppose that t_l exists over some universal domains K_i of characteristics p_i . Then t_l also exists over any (reasonable) S as in the previous assertion.

Proof. (1) This is immediate from Proposition 2.2.1(xiii) and Theorem 2.4.1(I).

(2) The previous assertion yields the following: it suffices to verify that $\mathcal{H}_{\mathbb{Q}_l}^{et}$ is conservative over any characteristic p field K (here p can be 0) if it is so over some universal domain K' of characteristic p . We should verify that if $\mathcal{H}_{\mathbb{Q}_l}^{et}(M) = 0$ for an $M \in \mathrm{Obj} \mathrm{DM}_c(K)$, then $M = 0$. We fix some M .

We note that any object (and morphism) in $\mathrm{DM}_c(K)$ is defined over some finitely generated subfield F of K see Proposition 2.2.1(xi). Besides, for any extension of fields the corresponding base change functor for $D_c^b Sh^{et}(-, \mathbb{Q}_l)$ is conservative (see Theorem 2.4.1(III)(1)). Hence we obtain that $\mathcal{H}_{\mathbb{Q}_l}^{et}(M_F) = 0$ for the corresponding $M_F \in \mathrm{DM}_c(\mathrm{Spec} F)$. Therefore, we may assume that $K \subset K'$; denote the corresponding morphism by b . We have $\mathcal{H}_{\mathbb{Q}_l}^{et}(b^*M) = 0$; hence $b^*M = 0$ and the conservativity of b^* (see Proposition 2.2.1(xii)) yields the result.

(3) Theorem 3.1.4 implies that it suffices to verify that t_l exists over any characteristic p field K (here p can be 0) if it exists over some universal domain K' of characteristic p . We prove this using an argument that is rather similar to the one above.

Again, in order to prove the existence of t_l it suffices to verify the orthogonality axiom and the existence of t_l -decompositions for the classes described in Definition 3.1.1(3).

Arguing as above, we obtain the following: if t_l exists over K' , it also exists over its subfield K .

We consider an algebraically closed $K \subset K'$. Our arguments along with Lemma 3.2.1 yield the following: if a t_l -decomposition of $Z_{K'}$ for a $Z \in \mathrm{Obj} \mathrm{DM}_c(K)$ exists (over K'), then a t_l -decomposition of $Z_U[d]$ exists over some smooth connected K -variety U of dimension d

(i.e., a t_l -decomposition $Z_1 \rightarrow Z_U[-d] \rightarrow Z_2$ of $Z_U[-d] = u^*Z[-d]$ exists in $\mathrm{DM}_c(U)$, for $u : U \rightarrow \mathrm{Spec} K$ being the structure morphism of U). Similarly, if we have a non-zero $h \in \mathrm{DM}_c(K)(M, N)$, $M \in \mathrm{DM}_c(K)^{t_l \leq 0}$, $N \in \mathrm{DM}_c(K)^{t_l \geq 1}$, then it vanishes over a certain U (since it vanishes over K').

We denote by $s : \mathrm{Spec} K \rightarrow U$ the embedding of some K -point of U into U . Then $h = s^*u^*h$; hence $h = 0$. Now, Lemma 3.1.3(5) yields that $u^*[d]$ is t -exact. Since it is also conservative, we obtain that it suffices to verify that $Z_i = u^*s^*Z_i$ (for $i = 1, 2$). Since $\mathcal{H}_{\mathbb{Q}_l}^{et}$ is conservative by the previous assertion, it suffices to verify that $\mathcal{H}_{\mathbb{Q}_l}^{et}(Z_i) = (u \circ s)^*\mathcal{H}_{\mathbb{Q}_l}^{et}(Z_i)$. Now it remains to note that $\mathcal{H}_{\mathbb{Q}_l}^{et}(Z_i)$ can be obtained by applying $u^*[d]$ to the t_l -decomposition of $\mathcal{H}_{\mathbb{Q}_l}^{et}(Z)$.

It remains to prove that t_l exists over K if it exists over its algebraic closure. Similarly to the reasoning above, we obtain the following: for any $Z \in \mathrm{Obj} \mathrm{DM}_c(K)$ there exists a finite extension F/K (of some degree $d > 0$) such that a t_l -decomposition of Z_F exists (in $\mathrm{DM}_c(F)$), and also any $h \in \mathrm{DM}_c(K)(M, N)$, $M \in \mathrm{DM}_c(K)^{t_l \leq 0}$, $N \in \mathrm{DM}_c(K)^{t_l \geq 1}$, vanishes over a certain F . Then Proposition 2.2.1(vii) yields the existence of t_l over K . Indeed, if $f : F \rightarrow K$ is the corresponding morphism, [Bei02] yields that the homomorphism $\mathrm{DM}_c(K)(M, N) \rightarrow \mathrm{DM}_c(F)(f^*M, f^*N)$ is injective. Besides, since f_* is t -exact (see Lemma 3.1.3(4)), we obtain that a t_l -decomposition exists for f_*f^*Z ; hence it exists for Z also (since $\mathrm{DM}_c(S)^{t_l \leq 0}$ and $\mathrm{DM}_c(S)^{t_l \geq 0}$ are idempotent complete). \square

We also recall Proposition 2.2.1(ii) here, which yields that $\mathrm{DM}_c(K_i)$ is the category $\mathrm{DM}_{\mathrm{gm}}(K_i)$ of Voevodsky's motives. Hence it suffices to verify the conservativity of $\mathcal{H}_{\mathbb{Q}_l}^{et}$ and the existence of t_l for the latter categories (cf. Remark 4.1.2).

COROLLARY 3.2.4. (1) Suppose that t_l exist over some universal domains K_i of certain characteristics p_i . Then for any S as in Proposition 3.2.3 Chow weight filtrations and spectral sequences for $\mathcal{H}_{\mathbb{Q}_l,0}^{et}$ over S can be lifted to $\mathrm{MM}(S)$.

(2) Suppose that t_l exists for $\mathrm{DM}_c(K)$, where K is some universal domain of characteristic p (p is either a prime or 0); let S be a very reasonable scheme of characteristic p . Then there exists a weight filtration for $\mathrm{MM}(S)$ with the corresponding functors $W_{\leq i, \mathrm{MM}}$ such that for any $i \in \mathbb{Z}$ we have $W_i \mathcal{H}_{\mathbb{Q}_l,0}^{et} \cong \mathcal{H}_{\mathbb{Q}_l}^{et} \circ W_{\leq i, \mathrm{MM}} \circ H_0^{t_l}$.

Proof. (1) This is immediate from Remark 1.3.3(2).

(2) By Theorem 2.5.4(II)(1), Chow weight spectral sequences degenerate (at E_2) for $H' = \mathcal{H}_{\mathbb{Q}_l,0}^{et}$. Hence Proposition 1.3.6(II) yields the degeneration of Chow-spectral sequences also for $H = H_0^{t_l}$. Therefore, Proposition 1.3.6(I) yields the existence of a weight filtration for $\mathrm{MM}(S) = \underline{H}t_l$ such that $W_i H_0^{t_l} \cong W_{\leq i, \mathrm{MM}} \circ H_0^{t_l}$. It remains to apply Proposition 1.3.2(I). \square

Remark 3.2.5. (1) Certainly, the assumptions of the corollary also yield that for any open embedding $j : U \rightarrow S$ one can lift $j_{!*}$ from $Sh_{\mathrm{per}}^{et}(-, \mathbb{Q}_l)$ to $\mathrm{MM}(-)$. In particular, if U is regular and dense in S , $j_{!*}\mathbb{Q}_U \in \mathrm{DM}_c(S)^{t_l=0}$ could be called the ‘intersection motif’ of S ; it corresponds to the \mathbb{Q}_l -adic étale intersection homology of S .

(2) Conversely to part 2 of the corollary, suppose that for some S there exists some weight filtration for $\mathrm{MM}(S)$ such that $H_i^{t_l}(\mathrm{Chow}(S))$ is of weight i (for any $i \in \mathbb{Z}$; this assumption can be reduced to the following ones: $\mathbb{Q}(j)$ is of weight $-2j$ for any $j \in \mathbb{Z}$, whereas $p_!$ respects weights in the corresponding sense for p being a projective morphism of schemes). Then for $H = H_0^{t_l}$ one can easily see that $T(H, -)$ degenerates at E_2 (cf. the proof of Theorem 2.5.4(II)(1)). Certainly, this yields Conjecture 2.5.3 in this case (see Proposition 1.3.6(II)(1)). We obtain a good reason to believe Conjecture 2.5.3 (for a general S).

(3) Instead of assuming that t_l exists over a universal domain K (of characteristic p), it suffices to assume that it exists over all members of a family K_i of fields such that any finitely generated L of characteristic p embeds into one of K_i . One may take algebraically closed fields of characteristic p such that their transcendence degrees are not bounded (by any natural number) for K_i .

3.3 Consequences of the existence of a nice motivic t -structure

We now derive certain consequences from the existence of a nice motivic t -structure for $\mathrm{DM}_c(S)$; we will need some of them below in order to make a certain inductive step.

PROPOSITION 3.3.1. *Suppose that a nice t_l exists over S and that $m \in \mathbb{Z}$, $M \in \mathrm{Obj} \mathrm{DM}_c(S)$. Then the following statements are fulfilled.*

- (I) (1) The category $\mathrm{MM}_m(S) = \mathrm{MM} \cap Hw_{\mathrm{Chow}}[m]$ is (abelian) semisimple.
- (2) If $M \in \mathrm{DM}_c(S)^{t_l=0}$, then it possesses an increasing filtration $W_{\leq r, \mathrm{MM}} M$, $r \in \mathbb{Z}$, whose j th factor belongs to $\mathrm{MM}_j(S)$ for any $j \in \mathbb{Z}$; this filtration is Ht_l -functorial in M .
- (3) $M \in \mathrm{DM}_c(S)_{w_{\mathrm{Chow}} \leq m}$ (respectively $M \in \mathrm{DM}_c(S)_{w_{\mathrm{Chow}} \geq m}$) if and only if for any $j \in \mathbb{Z}$ we have $(W_{m+j} H_j^{t_l})(M) = H_j^{t_l}(M)$ (respectively $(W_{m+j-1} H_j^{t_l})(M) = 0$).
- (4) $M \in \mathrm{DM}_c(S)_{w_{\mathrm{Chow}} \leq m}$ (respectively $M \in \mathrm{DM}_c(S)_{w_{\mathrm{Chow}} \geq m}$) if and only if for any $j \in \mathbb{Z}$ we have $(W_{m+j} H_{\mathbb{Q}_l, j}^{et})(M) = H_{\mathbb{Q}_l, j}^{et}(M)$ (respectively $(W_{m+j-1} H_{\mathbb{Q}_l, j}^{et})(M) = 0$).
- (5) If $M \in \mathrm{Obj} \mathrm{Chow}(S) (\subset \mathrm{Obj} \mathrm{DM}_c(S))$, then it can be decomposed into a direct sum of objects of $\mathrm{MM}_j(S)[-j]$; this decomposition is unique up to a (non-unique) isomorphism.
- (6) If $M \in \mathrm{Obj} \mathrm{MM}_m(S)$, then it can be decomposed as a direct sum of simple objects of $\mathrm{MM}_m(S)$; this decomposition is unique up to an isomorphism.
- (II) Let S be of finite type over $\mathrm{Spec} \mathbb{F}_p$ (for $p \neq l$); $M \in \mathrm{DM}_c(S)^{t_l=0}$. Consider the weights for $D_c^b Sh^{et}(S, \mathbb{Q}_l)$ defined in [BBD82, § 5] (cf. Proposition 2.5.1(I)). Then $M \in \mathrm{DM}_c(S)_{w_{\mathrm{Chow}} \leq m}$ (respectively $M \in \mathrm{DM}_c(S)_{w_{\mathrm{Chow}} \geq m}$) if and only if $\mathcal{H}_{\mathbb{Q}_l}^{et}(M)$ is of weight less (respectively greater) than or equal to m .

Proof. (I) (1)–(3), (5) These are immediate from Proposition 1.4.2(II).

(4) We note that $H_{\mathbb{Q}_l, j}^{et}(M) \cong H_{\mathbb{Q}_l, 0}^{et}(H_j^{t_l}(M))$. Applying assertion (I)(3), we may assume that $M \in \mathrm{DM}_c(S)^{t_l=0}$ and consider only $m = 0$. Then applying Proposition 1.3.6(II)(2)) for $F = H_{\mathbb{Q}_l, 0}^{et}$, we obtain the result (we also use Proposition 1.3.6(II)(2)) in order to relate the weight filtration for MM to Chow weight spectral sequences).

(6) This is immediate from the semisimplicity of MM_m .

(II) This is immediate from assertion (I)(5) along with Theorem 2.5.4(I). \square

Remark 3.3.2. (1) $\mathrm{MM}_m(S)$ could be called the category of *pure motives of weight m* (over S).

(2) Consider a category MS of ‘homological S -motives’ whose objects are $\mathrm{DM}_c(S)_{w_{\mathrm{Chow}}=0}$, and

$$MS(M, N) = \mathrm{Im}(\mathrm{DM}_c(S)(M, N)) \rightarrow \bigoplus_{m \in \mathbb{Z}} Sh_{\mathrm{per}}^{et}(S, \mathbb{Q}_l)(H_{\mathbb{Q}_l, m}^{et}(M), H_{\mathbb{Q}_l, m}^{et}(N)).$$

We conjecture that it is (anti)-isomorphic to the category $M(S)$ described in [CH00, Definition 5.9] (cf. [Bon14, Remark 2.1.2]).

We obtain that if a nice t_l exists over S , then MS is isomorphic to the direct sum of $\mathrm{MM}_m(S)$ (as additive categories). Hence, MS is semisimple (cf. [CH00, Theorem 5.13]); so it could also be called the category of ‘numerical motives’. It is also easily seen that for any $M \in \mathrm{DM}_c(S)_{w_{\mathrm{Chow}}=0}$ the kernel of the projection $\mathrm{MM}(Z, Z) \rightarrow MS(Z, Z)$ is a nilpotent ideal (cf. [CH00, Theorem 6.9]).

(3) So we have proved that $\mathcal{H}_{\mathbb{Q}_l}^{et}$ ‘strictly respects weights’ if S is of finite type over $\text{Spec } \mathbb{F}_p$; this is also true for $\tilde{\mathcal{H}}_{\mathbb{Q}_l}^{et}$ if S is of finite type over $\text{Spec } \mathbb{Q}$. In [Wil08] a similar statement was established unconditionally for *Artin–Tate* motives over number fields.

(4) Assume that $I \in \text{Obj } D_c^b Sh_{\text{per}}^{et}(S, \mathbb{Q}_l)$ is semisimple (i.e., that it is a direct sum of shifts of semisimple objects of $Sh_{\text{per}}^{et}(S, \mathbb{Q}_l)$) and that I is a retract of $I' = \mathcal{H}_{\mathbb{Q}_l}^{et}(N)$ for some $N \in \text{Obj } \text{DM}_c(S)$ (under our assumptions, this is easily seen to be equivalent to I being *semisimple of geometric origin* in the sense of [BBD82, § 6.2.4]). Then our assumptions yield that I is a retract of $\bigoplus_{n \in \mathbb{Z}} H_{\mathbb{Q}_l, 0}^{et}(M_n)[n]$ for some $M_n \in \text{Obj } \bigoplus_{m \in \mathbb{Z}} \text{MM}_m(S)$.

Indeed, it suffices to verify this statement for a simple $I \in \text{Obj } Sh_{\text{per}}^{et}(S, \mathbb{Q}_l)$; hence we can assume that $N \in \text{Obj } \text{MM}(S)$. Then we have morphisms $I \rightarrow H_{\mathbb{Q}_l, 0}^{et}(W_{\leq m, \text{MM}} N)$ for all $m \in \mathbb{Z}$. Since I is simple, these morphisms are either zero or embeddings; hence I is a retract of one of $H_{\mathbb{Q}_l, 0}^{et}(Gr_m^W(N))$.

3.4 Reducing the existence of a nice t_l to the universal domain case

We are now ready to prove our second main result.

THEOREM 3.4.1. *Suppose that for any point of a very reasonable scheme S the category $\text{DM}_c(K)$ possesses a nice t_l . Then the same is also true for $\text{DM}_c(S)$.*

Proof. By Theorem 3.1.4, t_l for $\text{DM}_c(S)$ exists. It remains to verify that t_l is transversal to w_{Chow} . For any (fixed) $M \in \text{DM}_c(S)^{t_l=0}$ and $m \in \mathbb{Z}$ we should verify the existence of a nice decomposition of M (see Definition 1.4.1).

We again apply the Noetherian induction, and assume that the statement is fulfilled over any proper closed subscheme of S .

Let K be a generic point of S . Since $j_K^*[-d]$ is t -exact and j_K^* is weight-exact, a nice choice of (2) (with the corresponding $m_K = m - d$) exists for $j_K^* M[-d]$ (in $\text{DM}_c(K)$; see Theorem 2.3.1(III)). By Theorem 2.3.1(III) and Lemma 3.2.1, there exists an open embedding $j : U \rightarrow S$ (U contains K) along with a nice choice

$$A \xrightarrow{f'} j^* M \xrightarrow{g'} B \quad (9)$$

of (2).

We verify that this choice can be lifted to a one for M . We apply $j_{!*}$ to (9). Since $j_{!*}$ preserves monomorphisms and epimorphisms (see Proposition 1.5.3(IV)(v)), we obtain a three-term complex as in (5) (i.e., $f = j_{!*} f'$ is monomorphic, and $g = j_{!*} g'$ is epimorphic). For $Z = S \setminus U$ the middle-term homology object H_{mid} of the complex obtained belongs to $i_*(\text{DM}_c(Z))$ by Proposition 1.5.3(IV)(iii). Since i_* is t - and weight-exact, the inductive assumption yields that a nice choice of (2) exists for H_{mid} . Now suppose that $j_{!*}(W_{\geq m+1, \text{MM}} j^* M) \in \text{DM}_c(S)_{w_{\text{Chow}} \geq m+1}$ and $j_{!*}(W_{\leq m, \text{MM}} j^* M) \in \text{DM}_c(S)_{w_{\text{Chow}} \leq m}$. Then we can choose ‘trivial’ nice decompositions for these objects; hence Proposition 1.4.2(I) would yield that a nice decomposition exists for $j_{!*} j^* M$. Now, applying Proposition 1.4.2(I) again along with Proposition 1.5.3(IV)(iv), we obtain that a nice decomposition exists for M also.

Hence it remains to verify that $j_{!*}$ maps $\text{DM}_c(U)^{t_l=0} \cap \text{DM}_c(U)_{w_{\text{Chow}} \geq m+1}$ into $\text{DM}_c(S)_{w_{\text{Chow}} \geq m+1}$, and maps $\text{DM}_c(U)^{t_l=0} \cap \text{DM}_c(U)_{w_{\text{Chow}} \leq m}$ into $\text{DM}_c(S)_{w_{\text{Chow}} \leq m}$.

We fix some $M \in \text{DM}_c(U)^{t_l=0} \cap \text{DM}_c(U)_{w_{\text{Chow}} \geq m+1}$ (respectively $M \in \text{DM}_c(U)^{t_l=0} \cap \text{DM}_c(U)_{w_{\text{Chow}} \leq m}$). Since $j^* j_{!*} M \cong M$, it suffices to verify that $i^! j_{!*} M \in \text{DM}_c(Z)_{w_{\text{Chow}} \geq m+1}$ (respectively $i^* j_{!*} M \in \text{DM}_c(Z)_{w_{\text{Chow}} \leq m}$).

The inductive assumption for Z reduces the latter fact to a certain calculation of weight filtrations for $H_{\mathbb{Q}_l, n}^{et}$ of the corresponding motives; see Proposition 3.3.1(I)(4). In this form the

statement follows immediately from Proposition 1.5.3(ii) and Theorem 2.5.4(II)(2) (along with Theorem 2.4.1(I)). \square

Remark 3.4.2. (1) Our arguments demonstrate that the notions of weight structure and of its transversality with t -structures are really important for the study of the ‘weight filtration’ of $\mathrm{DM}_c(S)^{t_l=0}$ (cf. § 4.1 below). Indeed, it seems that one cannot apply our gluing argument in the setting of filtered abelian categories (though possibly one could find a way to apply some of the corresponding arguments of [CH00] in our context).

(2) In contrast to the setting of the Theorem 3.1.4, we cannot prove the niceness of t when S is not very reasonable (without assuming Conjecture 2.5.3 for it). The problem is that the ‘weight-exactness’ of $j_{!*}$ does not follow from the (Noetherian) inductive assumption considered in the proof of the theorem. Indeed, let $S = \mathrm{Spec} \mathbb{Z}_{(p)}$ (for a prime $p \neq l$); then one can glue $t_l(\mathrm{Spec} \mathbb{F}_p)$ with any ‘shift’ of $t_l(\mathrm{Spec} \mathbb{Q})$ (here we assume that $t_l(\mathrm{Spec} \mathbb{F}_p)$ and $t_l(\mathrm{Spec} \mathbb{Q})$ exist, and consider $(\mathrm{DM}_c(\mathrm{Spec} \mathbb{Q})^{t' \leq 0}, \mathrm{DM}_c(\mathrm{Spec} \mathbb{Q})^{t' \geq 0}) = (\mathrm{DM}_c(\mathrm{Spec} \mathbb{Q})^{t_l \leq i}, \mathrm{DM}_c(\mathrm{Spec} \mathbb{Q})^{t_l \geq i})$ for any $i \in \mathbb{Z} \setminus \{0\}$). Then the niceness of t_l over $\mathrm{Spec} \mathbb{Q}$ is equivalent to the niceness of t' ; yet it seems highly improbable for $j_{!*}$ to be weight-exact for the weight structure obtained via this ‘shifted gluing’. Hence in order to control the niceness of t_l for S in this case, one needs some ‘extra’ information on it. It seems quite reasonable to control motives via their homology; to this end we have to extend Theorem 2.5.4(II) to this case (cf. some alternative arguments in [Wil12, § 7]).

We now prove that it suffices to verify the niceness of t_l over universal domains (only).

PROPOSITION 3.4.3. (1) *Suppose that a nice t_l exists over a universal domain K of characteristic $p > 0$. Then a nice t_l exists over any very reasonable $S/\mathrm{Spec} \mathbb{F}_p$.*

(2) *Suppose that t_l exists over the field of complex numbers. Then a nice t_l exists over any very reasonable $\mathrm{Spec} \mathbb{Q}$ -scheme.*

Proof. (1) The proof is rather similar to that of Corollary 3.2.4 (along with Proposition 3.2.3). We will only sketch it, outlining the differences.

Again, it suffices to verify that if the transversality property is fulfilled for motives over a field L , then it is fulfilled over any its algebraically closed subfield, and over its subfield K such that the extension L/K is algebraic.

Both of these statements can be proved using the arguments in the proof of Corollary 3.2.4. Indeed, by Proposition 1.4.2(III), we should verify that for any $M \in \mathrm{Obj} \mathrm{DM}_c(K)$ we have $(W_m H_0^{t_l})(M) \in \mathrm{DM}_c(K)_{w_{\mathrm{Chow}} \leq m}$ and $M/(W_m H_0^{t_l})(M) \in \mathrm{DM}_c(K)_{w_{\mathrm{Chow}} \geq m+1}$ (for all $m \in \mathbb{Z}$). This can be easily done by combining the arguments from the proof of Corollary 3.2.4(1) with Theorem 2.3.1(VI) (see also Remark 2.1.2(I)(1)); note that $f_* = f_!$ is weight-exact if f is a finite morphism.

(2) The statement is immediate from the previous assertion along with [Bei12, Proposition 1.5]. \square

Remark 3.4.4. (1) It is also easily seen that if t_l is nice over S , it is also nice over all of its subschemes and residue fields. Indeed, it suffices to note that for any open immersion i and (the complementary) closed embedding j the functors i_* and j^* are exact with respect to t_l and w_{Chow} , whereas i_* is a full embedding, j^* is a localization functor, and $\mathrm{Im} i_* = \mathrm{Ker} j^*$.

Certainly this observation is far from very exciting; yet it will make some of the formulations in § 4.2 nicer.

(2) [Bon12, Remark 2.1.4] describes a funny way to produce new examples of transversal weight and t -structures (out of ‘old’ ones for a triangulated $\underline{\mathcal{C}}$). To this end one should consider

the so-called ‘truncated categories’ \underline{C}_N (that are ‘usually’ defined for all $N \geq 0$). For our t, w , $\underline{C} = \mathrm{DM}_c(S)$, we have $\underline{C}_0 = K^b(\mathrm{Chow}(S))$. So (if certain ‘standard’ conjectures as listed in § 4.1 below hold) this category shares several nice properties with $\mathrm{DM}_c(S)$; this statement does not seem to be obvious.

4. Supplements

In § 4.1 we verify that the existence of t_l and its niceness (over a very reasonable scheme S) follow from certain (more or less) ‘standard’ motivic conjectures (over algebraically closed fields; here we use certain lists of those taken from [Bei02, § 1] and [Han99, § 2]).

In § 4.2 we note that our results yield a ‘motivic decomposition theorem’ (modulo the conjectures mentioned). In particular, we characterize pure motives over S in terms of those over its residue fields. This enables us to calculate $K_0(\mathrm{DM}_c(S))$.

In § 4.3 we extend our results from the case of $\mathrm{Spec} \mathbb{Z}[1/l]$ -schemes to the case of $\mathrm{Spec} \mathbb{Z}$ -schemes, and prove that the t -structure obtained does not depend on the choice of the corresponding l s. Here we need to assume that the numerical equivalence of cycles is equivalent to $\mathbb{Q}_{l'}$ -adic homological one (for any $l' \in \mathbb{P}$ and over universal domains of characteristic not equal to l' or 0).

4.1 Relating the existence of a (nice) t_l to ‘standard’ motivic conjectures

First we address the question which (more or less) ‘classical’ motivic conjectures ensure the existence of t_l over S , that is nice if S is a very reasonable scheme. By virtue of the results above, to this end it suffices to treat motives over universal domains only. So we consider motives over a universal domain K of characteristic $p \neq l$ (p is either a prime or 0); recall that $\mathrm{DM}_c(K) \cong \mathrm{DM}_{\mathrm{gm}}(K)$. None of the results of this subsection are essentially original (unless combined with some of our other results).

First we relate the existence of a nice t_l to (the widely believed to be true) conjectures A–C of [Bei02, § 1.2].

PROPOSITION 4.1.1. *The existence of a nice t_l for $\mathrm{DM}_{\mathrm{gm}}(K)$ is equivalent to the conjunction of the following statements.*

- (A) *The étale cohomology functor RH_{et} is conservative on Voevodsky’s $\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(K)$.*
- (B) *There exists a t -structure for $\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(K)$ such that RH_{et} is t -exact.*
- (C) *All the homology of the motives of smooth projective varieties with respect to this t -structure is semisimple.*

Proof. Since étale cohomology is (‘anti’)-compatible with Tate’s twists, the t -structure mentioned can be extended to the whole $\mathrm{DM}_{\mathrm{gm}}(K) = \mathrm{DM}_c(K)$; we will denote this extension by t_{MM} .

Now it is easy to see that $t_{\mathrm{MM}} = t_l$. Indeed, composing RH_{et} with Poincaré duality for $\mathrm{DM}_c(K)$, one obtains (a certain version of) étale homology for it, whereas conditions A and B together yield that t_{MM} is strictly compatible with $\mathcal{H}_{\mathbb{Q}_l}^{et}$ (as prescribed by our Definition 3.1.1). Note here that the Poincaré duality for $\mathrm{DM}_{\mathrm{gm}}(K)$ exists for K of any characteristic (by an argument of M. Levine described in [HK06, Appendix B]).

Next, condition C easily yields that the t_{MM} -homology objects for arbitrary Chow motives (‘in our sense’) are semisimple (here we apply the compatibility of $\mathcal{H}_{\mathbb{Q}_l}^{et}$ with Tate twists). Moreover, [Bei12, Proposition 1.4(ii)] yields the existence of the corresponding Chow–Kunneth decompositions (see Remark 1.4.3); hence the functors $H_i^{t_l}[-i]$ respect Chow motives. Thus t_l is

nice (over K) by Proposition 1.4.2(V) (since the condition $\text{Obj } \underline{A}_m \cap \text{Obj } \underline{A}_j = \{0\}$ for $m \neq j$ is immediate from the orthogonality axiom of weight structures).

The converse implication is even easier (and is of no great interest to us). \square

Remark 4.1.2. (1) Here and throughout this paper we use the following observation: although the author does not know whether all possible versions of the (\mathbb{Q}_l) -étale homology realization for motives over a field K are isomorphic, one can still be sure that all of them yield the same t_l . Indeed, we have spectral sequences $T_{w_{\text{Chow}}}(H_{\mathbb{Q}_l,0}^{\text{ét}}, -)$ (for any version of $H_{\mathbb{Q}_l,0}^{\text{ét}}$) that degenerate at E_2 (in this case; see Theorem 2.5.4(II)(1)). Hence $H_{\mathbb{Q}_l,m}^{\text{ét}}(M)$ vanishes (for some $m \in \mathbb{Z}$) if and only if $E_2^{p,m-p}(T) = 0$ for any $p \in \mathbb{Z}$. Now, in order to calculate $E_2^{p,m-p}(T)$ it suffices to know the restriction of $\mathcal{H}_{\mathbb{Q}_l}^{\text{ét}}$ to $\text{Chow}(K)$ (see Proposition 1.3.2(I)), and certainly the latter does not depend on the choice of the version for $\mathcal{H}_{\mathbb{Q}_l}^{\text{ét}}$.

(2) Combining Proposition 3.2.3(2) with this spectral sequence argument, we obtain that the conservativity of the étale realization of motives (over fields or over general $\mathbb{Z}[1/l]$ -base schemes) follows from the following conjecture: if K is an algebraically closed field of characteristic distinct from l , then any morphism of $\text{Chow}(K)$ -motives that yields an injection on their étale (co)homology, splits (cf. [Bon09, Proposition 7.4.2]). Note that the latter conjecture easily follows from the niceness of t_l over K (since t_l ‘splits’ Chow motives into direct sums of objects of semisimple categories $\text{MM}_m[-m]$, whereas $\mathcal{H}_{\mathbb{Q}_l}^{\text{ét}}$ is conservative on $\text{MM}_m[-m]$). Now, by virtue of the results below, the niceness of t_l follows from ‘standard’ conjectures. Hence, there is good reason to believe this (‘Chow-splitting’) conjecture.

It may be of interest to study the connection of our results with those of [Ayo07].

We now verify (briefly) the following statement.

PROPOSITION 4.1.3. *For K as above, assume that standard conjecture D of [Han99] (that \mathbb{Q}_l -homological equivalence of cycles coincides with numerical equivalence), and Murre’s conjectures A, D, and Van (as stated in [Han99, § 2]) are fulfilled. Then a nice t_l exists over K .*

Proof. First we note that Murre’s conjecture A yields the existence of Chow–Kunneth decompositions of motives of smooth projective varieties (over K), i.e., any such motif can be decomposed (in $\text{Chow}(K)$) into a direct sum of motives each of which has only one non-zero \mathbb{Q}_l -adic (co)homology group. Here and below we can consider $\mathcal{H}_{\mathbb{Q}_l}^{\text{ét}}$ instead of étale cohomology; cf. the proof of Proposition 4.1.1. Next, (the proof of) [Han99, Proposition 2.4] implies that the conjectures mentioned imply all the remaining Murre conjectures (we can apply [Han99, Proposition 2.4] here since the Lefschetz type standard conjecture B used in its proof follows from standard conjecture D by the main result of [Smi97]). We define $\text{MM}_m(K)$ as the subcategory of $\text{Chow}[m] \subset \text{DM}_{\text{gm}}(K)$ consisting of objects whose \mathbb{Q}_l -étale (co)homology is concentrated in degree 0.

Now [Han99, Proposition 2.3] yields that the categories $\text{MM}_m(K)$ are isomorphic to the corresponding pieces of the category of \mathbb{Q}_l -étale homological motives. Conjecture D embeds them into the category of numerical motives (which is semisimple by the main result of [Jan92]); hence they are semisimple also. Next, the arguments used for the proof of [Han99, Proposition 2.9] yield for $\text{MM}_m(K)$ the orthogonality conditions of Proposition 1.4.2(IV). Besides, by [Bon12, Lemma 1.1.1(6)] these conditions also yield that the category $C \subset \text{Chow}(K)$ with $\text{Obj } C = \bigoplus \text{Obj } \text{MM}_m(K)[-m]$ is idempotent complete; hence $C = \text{Chow}(K)$. Since $\langle \text{Chow}(K) \rangle = \text{DM}_{\text{gm}}(K)$ (see Proposition 2.2.1(ii)), Proposition 1.4.2(IV) yields that $\text{DM}_{\text{gm}}(K)$ possesses a t -structure t_{MM} that is transversal to w_{Chow} . Since all objects of $\underline{H}t_{\text{MM}}$ possess filtrations

whose factors belong to $\mathrm{MM}_m(K)$ (see Proposition 1.4.2(II)(4)), we obtain that $\mathrm{Obj} \, \underline{H}t_{\mathrm{MM}} \subset \mathrm{DM}_c(K)^{t_l=0}$; hence $\mathcal{H}_{\mathbb{Q}_l}^{et}$ is t -exact with respect to t_{MM} . Moreover, Murre's conjecture Van yields that $\mathcal{H}_{\mathbb{Q}_l}^{et}$ does not kill non-zero objects of $\mathrm{MM}_m(K)$ (since it does not kill non-zero Chow motives). Then Proposition 1.4.2(II)(4) also implies that $\mathrm{Obj} \, \underline{H}t_{\mathrm{MM}} = \mathrm{DM}_c(K)^{t_l=0}$; cf. the proof of Proposition 3.3.1(I)(4). Hence $t_{\mathrm{MM}} = t_l$ (see Remark 1.2.3(3)), and we obtain the result desired. \square

Remark 4.1.4. (1) Alternatively, one can prove the existence of the motivic t -structure for $\mathrm{DM}_{\mathrm{gm}}(K)$ using the arguments from the proof of [Han99, Theorem 3.4], whereas (the proof of) [Han99, Proposition 2.8] allows us to verify the conditions of Proposition 1.4.2(V) that ensure (in this case) that $w_{\mathrm{Chow}}(K)$ is transversal to t_l .

(2) For a characteristic 0 field K one can apply the results of Corti and Hanamura directly (after replacing étale cohomology by $\mathcal{H}_{\mathbb{Q}_l}^{et}$ using Poincaré duality). Indeed, it was proved in [Bon09, § 4] that in this case Hanamura's triangulated category of motives is isomorphic to $\mathrm{DM}_{\mathrm{gm}}(K)^{op}$.

(3) It seems that the existence of t_l without any additional assumptions does not imply its niceness (at least, easily) over positive characteristic fields (one also needs to assume the Hodge standard conjecture or the conjectures mentioned above for this matter). Over (very reasonable Spec \mathbb{Q} -schemes and) characteristic 0 fields one does not need any extra assumptions; cf. Proposition 3.4.3(2) (and also [Bei12, Proposition 2.2]).

4.2 Our ‘motivic decomposition theorem’

Our results easily yield a motivic version of the celebrated topological decomposition theorem (for perverse sheaves; see Remark 4.2.4). In particular, we characterize pure motives (see Remark 3.3.2(1)) ‘pointwise’. In order to formulate our results, we need a certain intermediate image functor for j_K , where $K \in \mathcal{S}$.

First let K be a generic point of S of dimension d (see § 2.1; so K is the spectrum of a field). Suppose that a nice t_l exists over S ; then it also exists over K (see Remark 3.4.4(1)). We define $j_{K!}^d$ for $M \in \mathrm{MM}_m(K)$ ($m \in \mathbb{Z}$) in the following way. First we lift $M[d]$ to a certain $M_U \in \mathrm{MM}_{m+d}(U)$ for some open $U \subset S$, $K \in U$; here we use Lemma 3.2.1 and Theorem 2.3.1(III) (cf. the proof of Theorem 3.4.1). M_U is semisimple in $\mathrm{MM}(U)$ (see Proposition 3.3.1(I)(6)); and we take M'_U as the sum of those components of M_U that are not killed by j_K^{U*} (for the corresponding morphism $j_K^U : K \rightarrow U$; note that M'_U is determined by M_U uniquely up to an isomorphism). Finally, we set $j_{K!}^d M = j_! M'_U$, where $j : U \rightarrow S$ is the corresponding open immersion.

Now let K be an arbitrary point of S of dimension d , whose closure is $Z \subset S$; $i : Z \rightarrow S$ is the corresponding embedding. Then we lift $M[d]$ to $\mathrm{MM}_{m+d}(Z)$ using the procedure described above, and apply i_* in order to obtain $j_{K!}^d M$. Here we use the t_l - and w_{Chow} -exactness of $i_* = i_!$. Note, furthermore, that if we denote the composite immersion $U \rightarrow Z \rightarrow S$ by j , then we would have

$$j_{K!}^d M = \mathrm{Im}(H_0^{t_l} j_! M'_U \rightarrow H_0^{t_l} j_* M'_U) \quad (10)$$

in this case (also).

LEMMA 4.2.1. (1) $j_{K!}^d M$ does not depend on any choices (if a nice t_l exists over S). Moreover, $j_{K!}^d$ yields a full embedding (of categories).

(2) $j_{K!}^d M$ is functorially characterized by the following condition: it is a semisimple lift of M to $\mathrm{MM}_{m+d}(S)$ none of whose direct summands are killed by j_K^* .

Proof. This is an easy consequence of Proposition 1.5.3(IV)(vii)–(ix). \square

Remark 4.2.2. Alternatively, one could try to apply here the (somewhat parallel) arguments of [Sch12, §5]. However, some adjustments (along with certain results of [Bon14, §2.3]) are needed to do so.

PROPOSITION 4.2.3. *Let a nice t_l exist over S . Then for any $m \in \mathbb{Z}$ any object of $\mathrm{MM}_m(S)$ can be decomposed as a direct sum of $j_{K_i!}^{d_i} M_i$ for $K_i \in \mathcal{S}$ of dimension d_i , and $M_i \in \mathrm{MM}_{m-d_i}(K_i)$ being indecomposable objects. This decomposition is unique up to an isomorphism. Moreover, $M_i \cong H_{-d_i}^{t_l}(j_{K_i}^* M)$, whereas K_i can be characterized by the condition that $H_{-d_i}^{t_l}(j_{K_i}^* M) \neq 0$.*

Proof. We verify that M can be decomposed into a direct sum of some $j_{K_i!}^{d_i} M_i$ in some way. Since $\mathrm{MM}_m(S)$ is semisimple, it suffices to prove the following: if M is indecomposable, then it can be presented as $j_{K_M!}^{d_M} M_K$ for some $K_M \in \mathcal{S}$ of dimension d_M and $M_K \in \mathrm{MM}_{m-d}(K_M)$. We prove this statement by Noetherian induction (applying Remark 3.4.4(1) again).

Let K be a generic point of S . If $j_K^* M \neq 0$, Lemma 4.2.1 immediately implies that we can take $K_M = K$, $M_K = j_K^* M[-d]$. On the other hand, if $j_K^* M = 0$, then there exists an open immersion $j : U \rightarrow S$ ($K \in U$) such that $j^* M = 0$. Hence for the complementary closed embedding $i : Z \rightarrow S$ there exists a (simple) $M_Z \in \mathrm{MM}_m(Z)$ such that $M \cong i_* M_Z$ (since i_* is Chow weight and t -exact). Hence it suffices to apply the inductive assumption to M_Z (see (10)).

It remains to verify that for $K, K' \in \mathcal{S}$ of dimensions d and d' respectively, and $M \in \mathrm{MM}_n(K)$ ($n \in \mathbb{Z}$), we have $H_{-d'}^{t_l}(j_{K'}^* j_{K!}^d(M)) = 0$ if $K' \neq K$ and $= M$ otherwise. We consider three cases here: (1) $K' = K$, (2) K' belongs to the closure Z of K in S , and (3) K' does not belong to Z .

Denote the embedding of Z into S by i ; denote the complementary immersion by j and the morphism $K \rightarrow Z$ by j_K^Z . In case (1) it suffices to note that $j_{K!}^{Z,d} M$ is a lift of $M[d]$ to $\mathrm{DM}_c(Z)$, whereas $i^* i_* = 1_{\mathrm{DM}_c(Z)}$. In case (3) it suffices to note that $j_{K'}^*$ factorizes through j and that $j^* i_* = 0$. In case (2) we can assume that $Z = S$ (since $i^* i_* = 1_{\mathrm{DM}_c(Z)}$); then our claim easily follows from Proposition 1.5.3(IV)(ii). \square

Remark 4.2.4. (1) The ‘usual’ topological decomposition theorem (see [CH00, Theorem 5.7]) states (for S a variety over a field) that if $X \rightarrow S$ is a proper morphism, and X is regular, then $f_* \mathbb{Q}_{lX} \in \mathrm{Obj} D_c^b Sh^{et}(S, \mathbb{Q}_l)$ splits as a direct sum of its t -homology, whereas its homology (perverse) sheaves can be presented as direct sums of intermediate images of pure \mathbb{Q}_l -local systems supported on some subvarieties of S . We verify that this decomposition can be lifted to $\mathrm{DM}_c(S)$ (hence, we can improve [CH00, Theorem 5.14]) even if we replace \mathbb{Q}_{lX} here by $K = \mathcal{H}_{\mathbb{Q}_l}^{et}(N)$ for some $N \in H_{\mathbb{Q}_l,0}^{et}(\mathrm{MM}_n(X))$ (for some $n \in \mathbb{Z}$) and do not require X to be regular.

We note that $f_* K = \mathcal{H}_{\mathbb{Q}_l}^{et}(f_* N)$ (see Theorem 2.4.1(I)), whereas $f_* N \in \mathrm{DM}_c(S)_{w_{\mathrm{Chow}}=n}$ (see Theorem 2.3.1(II)(1)). By Proposition 3.3.1(I)(5) we obtain that t_l splits $f_* N$ into a direct sum of objects of $\mathrm{MM}_j(S)[n-j]$ (this statement is the relative generalization of the existence of Chow–Kunneth decompositions).

Hence in order to achieve our goal it suffices to verify (by virtue of Proposition 4.2.3; for an $m \in \mathbb{Z}$) for any $M \in \mathrm{Obj} \mathrm{MM}_m(S)$ that the (perverse) homology of the corresponding $j_{K_i!}^{d_i} M_i$ can be presented as the intermediate image of a \mathbb{Q}_l -local system supported on some regular connected subvariety U_i of S , whereas K_i is the generic point of U_i (cf. [BBD82, Theorem 4.3.1(ii)]). As we have verified above, for any such U_i the motif $j_{K_i!}^{d_i} M_i$ can be presented as $j_{U_i!} M_{U_i}$ for some $M_{U_i} \in \mathrm{Obj} \mathrm{MM}_m(U_i)$ (here we denote by $j_{U_i!}$ the composite of the intermediate image functor for the embedding of U_i into its closure Z_i with the direct image $\mathrm{DM}_c(Z_i) \rightarrow \mathrm{DM}_c(S)$). We should prove that we can choose U_i, M_{U_i} such that $\mathcal{H}_{\mathbb{Q}_l}^{et}(M_{U_i}) \in \mathrm{Obj} Sh_{\mathrm{per}}^{et}(S, \mathbb{Q}_l)$ is a local \mathbb{Q}_l -system on U_i .

By Theorem 2.3.1(III)(2), we can assume (if we choose U_i to be small enough) that there exist a regular scheme U'_i , a finite universal homeomorphism $g : U'_i \rightarrow U_i$, a smooth projective morphism $h : P_i \rightarrow U'_i$, and an $s \in \mathbb{Z}$ such that M_{U_i} is a retract of $(g \circ h)_* \mathbb{Q}_P(s)[2s + m]$. It remains to note that the homology sheaves of $(g \circ h)_* \mathbb{Q}_{P_i}(s) \in \text{Obj } D_c^b Sh^{et}(U_i, \mathbb{Q}_l)$ are pure local systems (since this is true for the ‘canonical’ homology of this derived category, and U_i is regular, we obtain that the perverse homology equals the canonical homology).

(2) More generally, assume that I is a retract of an object I' of $\mathcal{H}_{\mathbb{Q}_l}^{et}(\bigoplus_{m,j \in \mathbb{Z}} \text{MM}_m(X)[j])$. As noted in Remark 3.3.2(4), this condition is fulfilled (in particular) if I is a semisimple complex of geometric origin (in the sense of [BBD82]). Then $f_* I'$ belongs to $\mathcal{H}_{\mathbb{Q}_l}^{et}(\bigoplus_{m,j \in \mathbb{Z}} \text{MM}_m(S)[j])$. Since local systems over subschemes of S yield Krull–Schmidt subcategories, we obtain that $f_* I$ can be presented as a direct sum of retracts of $\mathcal{H}_{\mathbb{Q}_l}^{et}(j_{U_i!} M_{U_i}[s_i])$ for (U_i, M_{U_i}, s_i) corresponding to $f_* I'$.

Thus we obtain a certain motivic analogue of [BBD82, Theorem 6.2.5]. Yet note that (in contrast to [BBD82]) f_* does not preserve semisimplicity of perverse sheaves in general. Indeed, even if we take $S = \text{Spec } K$ for a (general) field K and a smooth projective X/K , then the étale cohomology of $X_{K^{\text{sep}}}$ need not be semisimple as $\text{Gal}(K)$ -representations (for example, for $K = \mathbb{Q}_p$ we do not have semisimplicity for H_{et}^1 of an elliptic curve with split multiplicative reduction; see [Sil94, Exercise 5.13]).

We are now (also) able to calculate $K_0(\text{DM}_c(S))$.

COROLLARY 4.2.5. *Define $K_0(\text{DM}_c(S))$ as the abelian group whose generators are $[C], C \in \text{Obj DM}_c(S)$; if $D \rightarrow B \rightarrow C \rightarrow D[1]$ is a distinguished triangle in $\text{DM}_c(S)$ then we set $[B] = [C] + [D]$. Let a nice t_l exist over S . Then $K_0(\text{DM}_c(S))$ is a free abelian group with a basis indexed by isomorphism classes of indecomposable objects of $\text{MM}_m(K_i)$ for K_i running through all elements of \mathcal{S} , m running through all integers.*

Proof. By [Bon12, Proposition 1.2.6], $K_0(\text{DM}_c(S))$ is a free abelian group with a basis indexed by isomorphism classes of indecomposable objects of $\text{MM}_m(S)$ (for m running through all integers). Now the result follows easily from Proposition 4.2.3. \square

4.3 Changing l ; the case of $\text{Spec } \mathbb{Z}$ -schemes

First we study the question when $t_{l'}$ exists and coincides with t_l for prime $l \neq l'$ (we fix the primes, and define $t_{l'}$ similarly to t_l).

PROPOSITION 4.3.1. (1) *Suppose that $\mathcal{H}_{\mathbb{Q}_l}^{et}$ and $\mathcal{H}_{\mathbb{Q}_{l'}}^{et}$ exist and coincide over universal domains of all characteristics not equal to l, l' . Then they also exist and coincide over any reasonable $\text{Spec } \mathbb{Z}[1/l, 1/l']$ -scheme S .*

In particular, this assertion holds if $\mathcal{H}_{\mathbb{Q}_l}^{et}$ and $\mathcal{H}_{\mathbb{Q}_{l'}}^{et}$ are nice over the universal domains mentioned.

(2) *Suppose that for any prime $p \neq l, l'$ there exists a universal domain K of characteristic p such that there exists a nice t_l for $\text{DM}_c(K)$, and $\mathcal{H}_{\mathbb{Q}_{l'}}^{et}$ -homological equivalence of cycles coincides with the numerical equivalence one over K . Then t_l and $t_{l'}$ exist and coincide on $\text{DM}_c(S)$ for any reasonable $\text{Spec } \mathbb{Z}[1/l, 1/l']$ -scheme S .*

Proof. (1) First we note that the niceness of t_l and $t_{l'}$ for $\text{DM}_c(K)$ yields that they coincide (on $\text{DM}_c(K)$) by [Bei02, Proposition 4.5]. So, it remains to verify that t_l and $t_{l'}$ exist and coincide over S if this is true over universal domains (of all characteristics not equal to l, l').

By Corollary 3.2.4(1), t_l and $t_{l'}$ exist for $\mathrm{DM}_c(S)$. By Lemma 3.1.3(7), it suffices to verify that $\mathrm{DM}_c(F)^{t_{l'} \leq 0} = \mathrm{DM}_c(F)^{t_l \leq 0}$ and $\mathrm{DM}_c(F)^{t_{l'} \geq 0} = \mathrm{DM}_c(F)^{t_l \geq 0}$ for any point F of S . An argument similar to the one used in the proof of Corollary 3.2.4(1) yields that we can replace F by one of our universal domains (of the same characteristic).

(2) By the previous assertion, it suffices to verify that t_l coincides with $t_{l'}$ over K .

Since all $\mathrm{MM}_i(K) = \mathrm{DM}_c(K)^{t_l=0} \cap \mathrm{DM}_c(K)_{w_{\mathrm{Chow}}=i}$ are semisimple, we obtain that $\mathcal{H}_{\mathbb{Q}_l}^{et}$ -homological equivalence of cycles is equivalent to numerical equivalence (over K). Indeed, consider the functor $\oplus Gr_m^{t_l} : \mathrm{Chow}(K) \rightarrow \oplus \mathrm{MM}_m(K)$ that is given by the direct sum of all (shifted) t_l -homology of Chow motives; see Remark 3.3.2(2). It kills exactly those morphisms of Chow motives that are \mathbb{Q}_l -homologically equivalent to zero as cycles (since $\mathcal{H}_{\mathbb{Q}_l}^{et}$ does not kill non-zero morphisms in $\mathrm{MM}_m(K)$). It remains to note that \mathbb{Q}_l -homological equivalence is finer than the numerical equivalence, and the category of numerical motives (over K) is semisimple (see [Jan92]). Besides, $\mathcal{H}_{\mathbb{Q}_{l'}}^{et}$ is also conservative on $\mathrm{MM}_m(K)$ for any $m \in \mathbb{Z}$.

Now, for a smooth projective P/K , [Kle94, Proposition 5.4] implies (in our setting) that the (Chow-)Kunneth decomposition of the motif of P coming from t_l is also its Kunneth decomposition with respect to $\mathcal{H}_{\mathbb{Q}_{l'}}^{et}$. Indeed, by [Kle94, Proposition 5.4] the numerical equivalence classes of the corresponding projectors do not depend on the choice of a Weil (co)homology theory; note that (by the main result of [Smi97]) we can replace the Hodge standard conjecture by standard conjecture D (see § 4.1) in the assumptions of [Kle94]. Hence $\mathcal{H}_{\mathbb{Q}_{l'}}^{et}$ sends $\mathrm{MM}_m(K)$ into $D_c^b Sh^{et}(K, \mathbb{Q}_{l'})^{t=0}$. Then Proposition 3.3.1(I)(2) easily yields that $\mathrm{DM}_c(K)^{t_{l'}=0} = \mathrm{DM}_c(K)^{t_l=0}$; boundedness yields that $t_{l'}$ coincides with t_l (see Remark 1.2.3(3)). \square

Now suppose that S is not (necessarily) a $\mathrm{Spec} \mathbb{Z}[1/l]$ -scheme. We note that $\mathcal{H}_{\mathbb{Q}_l}^{et}$ kills all motives ‘supported on’ $S \times_{\mathrm{Spec} \mathbb{Z}} \mathrm{Spec} \mathbb{F}_p$. In order to overcome this difficulty we introduce the following definition.

DEFINITION 4.3.2. Let $l \neq l'$ be primes. Consider the class $\mathrm{DM}_c(S)^{t_{l,l'} \leq 0}$ (respectively $\mathrm{DM}_c(S)^{t_{l,l'} \geq 0}$) consisting of those $M \in \mathrm{Obj} \mathrm{DM}_c(S)$ that satisfy $\mathcal{H}_{\mathbb{Q}_l}^{et}(M) \in D_c^b Sh^{et}(S[1/l], \mathbb{Q}_l)^{t_l \leq 0}$ and $\mathcal{H}_{\mathbb{Q}_{l'}}^{et}(M) \in D_c^b Sh^{et}(S[1/l'], \mathbb{Q}_{l'})^{t_{l'} \leq 0}$ (respectively $\mathcal{H}_{\mathbb{Q}_l}^{et}(M) \in D_c^b Sh^{et}(S[1/l], \mathbb{Q}_l)^{t_l \geq 0}$ and $\mathcal{H}_{\mathbb{Q}_{l'}}^{et}(M) \in D_c^b Sh^{et}(S[1/l'], \mathbb{Q}_{l'})^{t_{l'} \geq 0}$).

If $(\mathrm{DM}_c(S)^{t_{l,l'} \leq 0}, \mathrm{DM}_c(S)^{t_{l,l'} \geq 0})$ yield a t -structure for $\mathrm{DM}_c(S)$, we will say that (the t -structure) $t_{l,l'}$ exists over S .

Now we observe that ‘standard’ conjectures imply that $t_{l,l'}$ exists over any (reasonable) S and does not depend on l . We formulate a concise result of this kind here; further (somewhat stronger) statements of this sort can also be easily proved.

PROPOSITION 4.3.3. *Let l, l' be as above.*

(1) *Suppose that a nice t_l exists over any universal domain of any characteristic not equal to l , a nice $t_{l'}$ exists over a universal domain of characteristic l , and $\mathcal{H}_{\mathbb{Q}_{l'}}^{et}$ -homological equivalence of cycles is equivalent to numerical equivalence over any universal domain of any characteristic not equal to l' or 0. Then $t_{l,l'}$ is a t -structure over any (reasonable) S .*

(2) *Suppose, moreover, that for any prime p (distinct from l, l'), $\mathcal{H}_{\mathbb{Q}_p}^{et}$ -homological equivalence of cycles coincides with their numerical equivalence over any universal domain of any characteristic $\neq p$. Then $t_{l,l'}$ does not depend on the choice of the pair l, l' .*

Proof. (1) It is easy to see that $t_{l,l'}$ can be characterized similarly to Lemma 3.1.3(7), i.e., $M \in \mathrm{DM}_c(S)^{t_{l,l'} \leq 0}$ (respectively $M \in \mathrm{DM}_c(S)^{t_{l,l'} \geq 0}$) if and only if for any $K \in \mathcal{S}$, for K of dimension

d , we have $j_K^* M[-d] \in \mathrm{DM}_c(K)^{t_{l,l'} \leq 0}$ (respectively $j_K^! M[-d] \in \mathrm{DM}_c(K)^{t_{l,l'} \geq 0}$). Indeed, this is an easy consequence of Lemma 3.1.3(7).

Next we note that over characteristic 0 universal domains the homological equivalence of cycles relation does not depend on the choice of l since it can be described in terms of singular (co)homology. Hence Proposition 4.3.1 yields that $t_l = t_{l'}$ over any field of characteristic not equal to l, l' .

It easily follows that $t_{l,l'}(S)$ can be glued from t_l for $S[1/l]$ and from $t_{l'}$ for $S \times_{\mathrm{Spec} \mathbb{Z}} \mathrm{Spec} \mathbb{F}_l$; see Propositions 1.5.3(II)(1) and 2.2.1(viii).

(2) It suffices to apply Proposition 4.3.1 (again) and the arguments described above. \square

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